Eventually dendric sets

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joint work with Dominique Perrin

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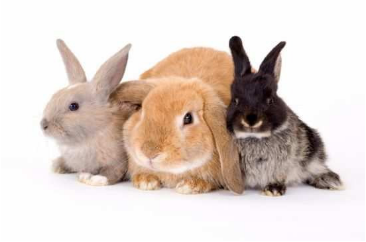
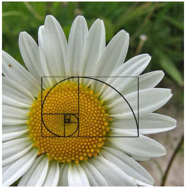
Fibonacci

$$\mathbf{x} = abaababaabaababa\cdots$$

$$\mathbf{x} = \lim_{n\to\infty} \varphi^n(a) \qquad \text{where} \qquad \varphi : \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$





Fibonacci

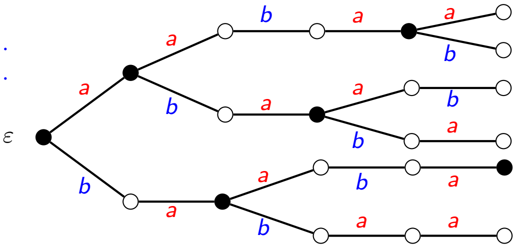
$$\mathbf{x} = abaababaabaababa\cdots$$

The *Fibonacci set* (set of factors of $\mathbf{x}$) is a Sturmian set.

Definition

A *Sturmian* set $S \subset \mathcal{A}^*$ is a factorial set such that $p_n = \mathrm{Card}\,(S \cap \mathcal{A}^n) = n + 1$.

n :	0	1	2	3	4	5	$\cdots$
p_n :	1	2	3	4	5	6	$\cdots$



2-coded Fibonacci

$$\mathbf{x} = ab\ aa\ ba\ ba\ ab\ aa\ ba\ ba\ \cdots$$

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$$f : \begin{cases} u & \mapsto & aa \\ v & \mapsto & ab \\ w & \mapsto & ba \end{cases}$$

2-coded Fibonacci

$$\mathbf{x} = ab\ aa\ ba\ ba\ ab\ aa\ ba\ ba\ \cdots$$

$$f^{-1}(\mathbf{x}) = v\ u\ w\ w\ v\ u\ w\ w\ \cdots$$

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Arnoux-Rauzy sets

Definition

An *Arnoux-Rauzy* set is a factorial set closed by reversal with $p_n = (\mathrm{Card}\,(\mathcal{A}) - 1)n + 1$ having a unique right special factor for each length.

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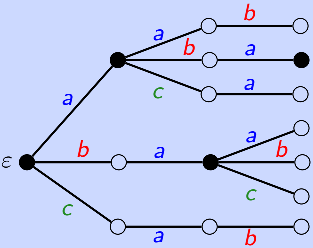
Example (Tribonacci)

Factors of the fixed point $\psi^\omega(a)$ of the morphism $\psi : a \mapsto ab, \quad b \mapsto ac, \quad c \mapsto a.$



n :	0	1	2	3	$\cdots$
p_n :	1	3	5	7	$\cdots$

$$p_n = 2n + 1$$

2-coded Fibonacci

$$f^{-1}(\mathbf{x}) = \ v\ u\ w\ w\ v\ u\ w\ w\ \cdots$$

Is the set of factors of $f^{-1}(S)$ an Arnoux-Rauzy set ?

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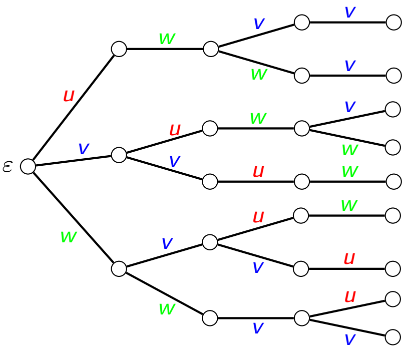

$$p_n = 2n + 1$$

n :	0	1	2	3	4	$\cdots$
p_n :	1	3	5	7	9	$\cdots$

2-coded Fibonacci

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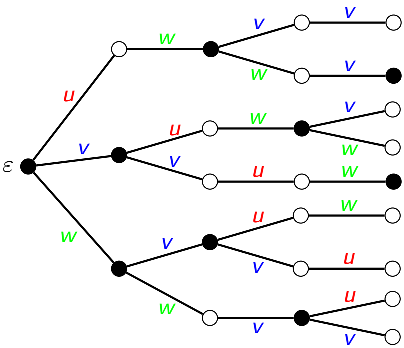
Is the set of factors of $f^{-1}(S)$ an Arnoux-Rauzy set ? No !



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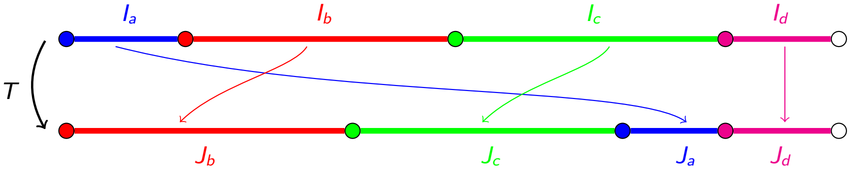
Interval exchanges

Let $(I_\alpha)_{\alpha \in \mathcal{A}}$ and $(J_\alpha)_{\alpha \in \mathcal{A}}$ be two partitions of $[0,1[$.
An *interval exchange transformation* (IET) is a map $T : [0,1[\to [0,1[$ defined by

$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha .$$



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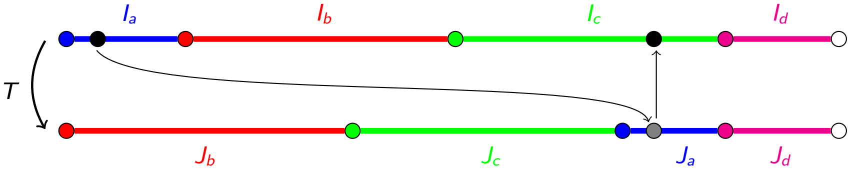

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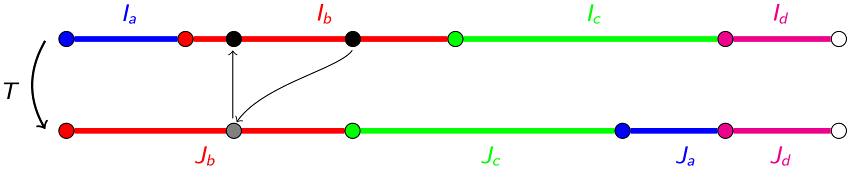

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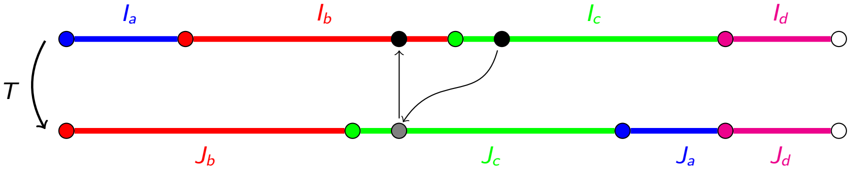

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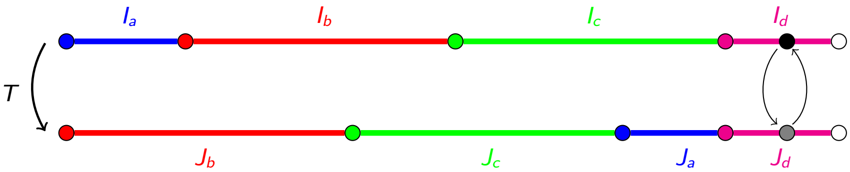

Interval exchanges

T is said to be *minimal* if for any point $z \in [0,1[$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[0,1[$.

T is said *regular* if the orbits of the non-zero separation points are infinite and disjoint.

Theorem [Keane (1975)]

A regular interval exchange transformation is minimal.

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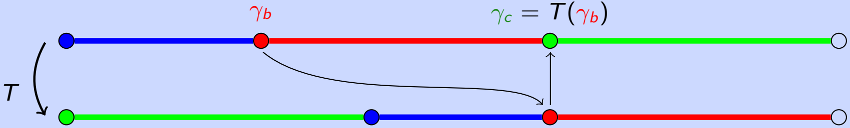
Example (the converse is not true)

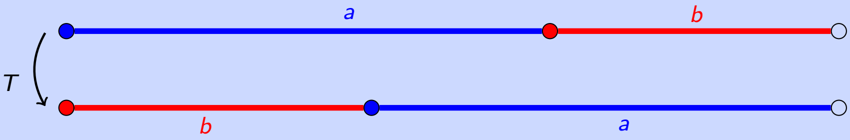


Interval exchanges

The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in \mathcal{A}^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha.$$

Example (Fibonacci, $z = (3-\sqrt{5})/2$)



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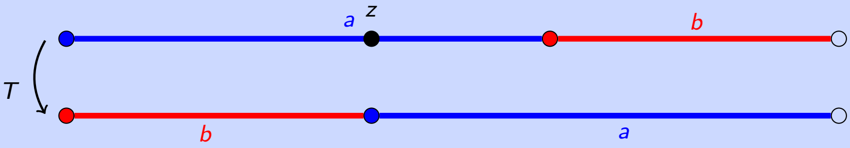

$$\Sigma_T(z) = a$$

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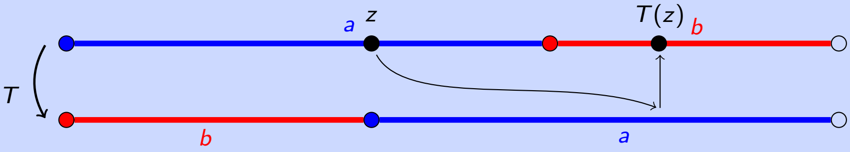

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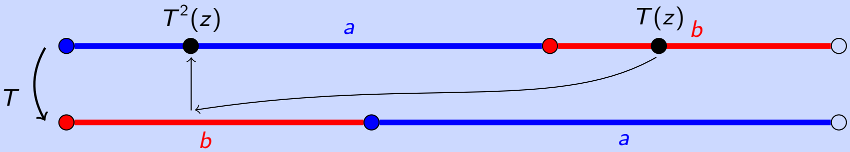

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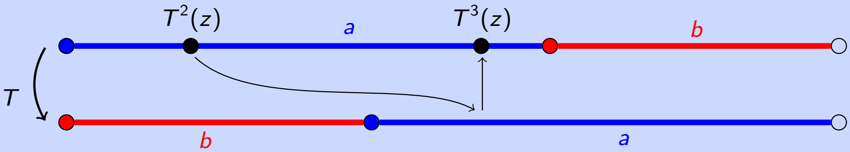

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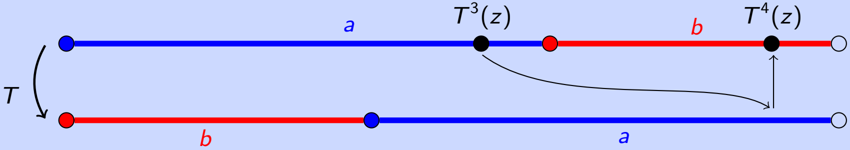

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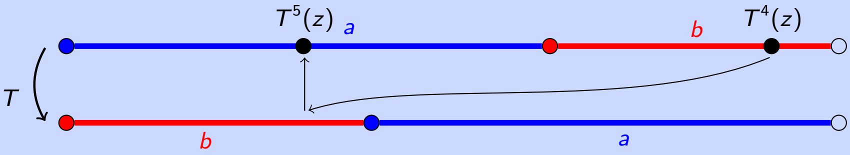

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Interval exchanges

The set $\mathcal{L}(T) = \bigcup_{z \in [0,1[} \mathrm{Fac}\,(\Sigma_T(z))$ is said a (*minimal*, *regular*) *interval exchange set*.

Remark. If T is minimal, $\mathrm{Fac}\,(\Sigma_T(z))$ does not depend on the point z.

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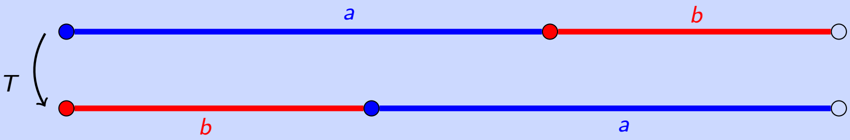

$$\mathcal{L}(T) = \left\{\varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab\ \ldots\right\}$$

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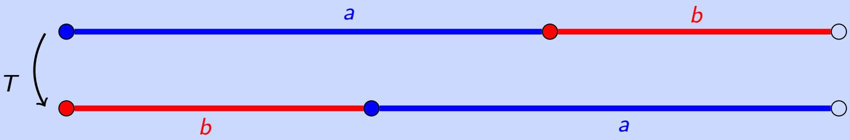
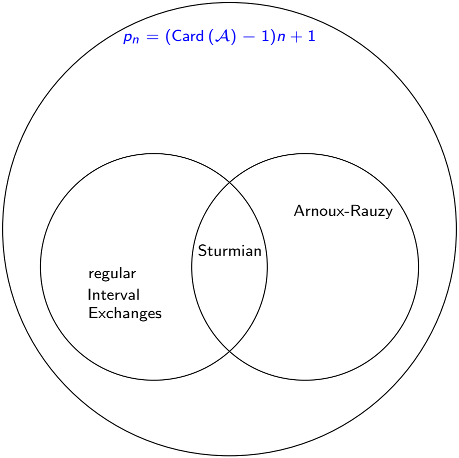


$$\mathcal{L}(T) = \left\{\varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab\ \dots\right\}$$

Proposition

Regular interval exchange sets have factor complexity $p_n = (\mathrm{Card}(\mathcal{A}) - 1)n + 1$.

Arnoux-Rauzy and Interval exchanges



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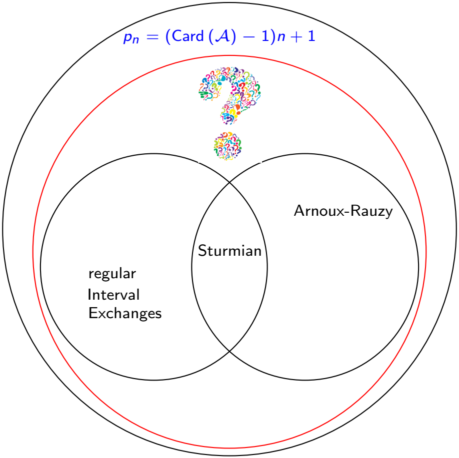

Extension graphs

The *extension graph* of a word $w \in S$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{a \in \mathcal{A} \mid aw \in S\}, \\ R(w) &= \{a \in \mathcal{A} \mid wa \in S\}, \\ B(w) &= \{(a,b) \in \mathcal{A} \times \mathcal{A} \mid awb \in S.\} \end{aligned}$$

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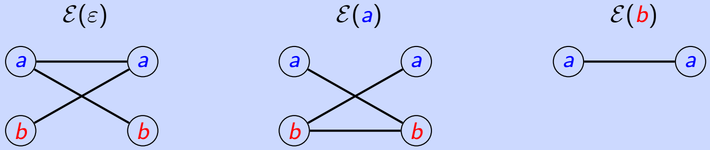

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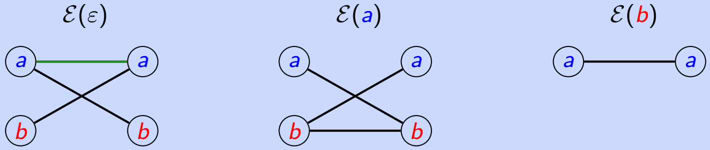

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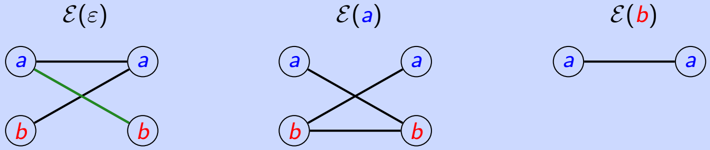

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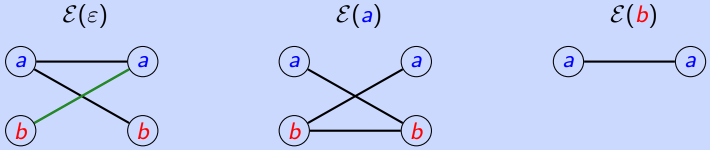

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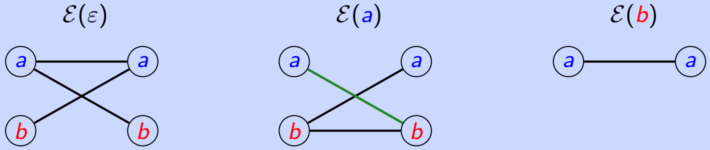

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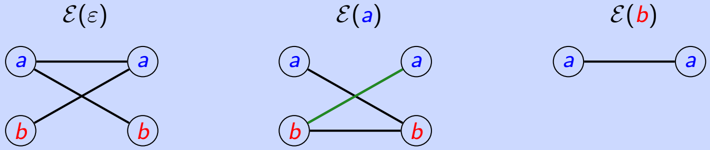

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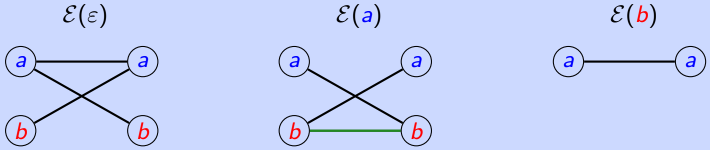

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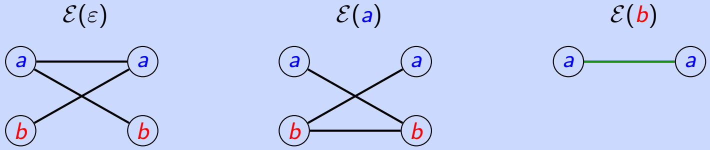

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The *multiplicity* of a word w is the quantity

$$m(w) = \mathrm{Card}\,(B(w)) - \mathrm{Card}\,(L(w)) - \mathrm{Card}\,(R(w)) + 1.$$

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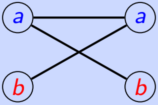
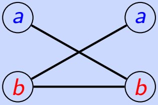
$\mathcal{E}(\varepsilon)$



$\mathcal{E}(a)$



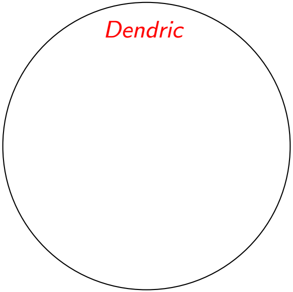
$\mathcal{E}(b)$

$m(a) = 3 - 2 - 2 + 1 = 0$

Dendric and neutral sets

Definition

A biextendable factorial set S is called a *dendric set* if the graph $\mathcal{E}(w)$ is a tree for any $w \in S$.



Dendric and neutral sets

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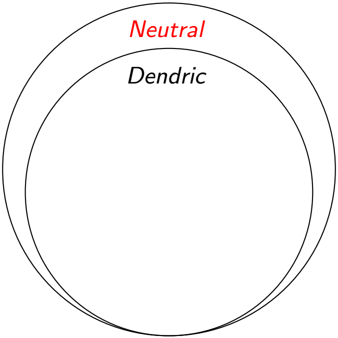
A biextendable factorial set S is called a *dendric set* if the graph $\mathcal{E}(w)$ is a tree for any $w \in S$. It is called *neutral* if every word w has multiplicity $m(w) = 0$.



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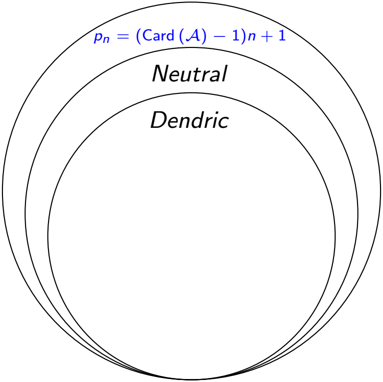

[*using* Cassaigne : "**Complexité et facteurs spéciaux**" (1997).]

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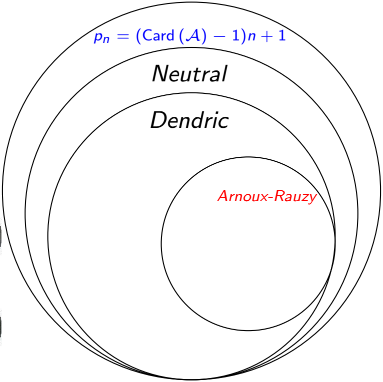

[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone : "**Acyclic, connected and tree sets**" (2014).]

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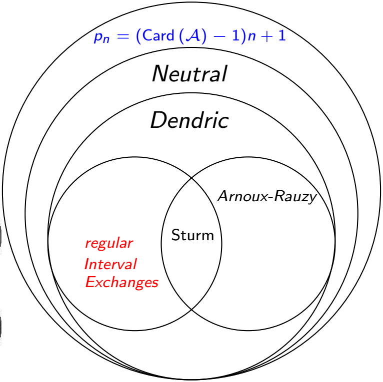


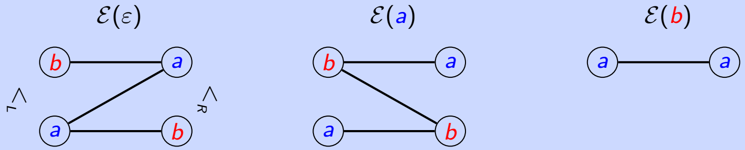
[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone : "**Bifix codes and interval exchanges**" (2015).]

Planar dendric sets

Let $<_L$ and $<_R$ be two orders on $\mathcal{A}$.
For a set S and a word $w \in S$, the graph $\mathcal{E}(w)$ is *compatible* with $<_L$ and $<_R$ if for any $(a, b), (c, d) \in B(w)$, one has

$$a <_L c \quad \Longrightarrow \quad b \leq_R d.$$

Example (Fibonacci, $b <_L a$ and $a <_R b$)

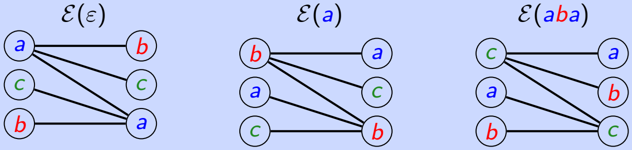


A biextendable set S is a *planar dendric set* w.r.t. $<_L$ and $<_R$ on $\mathcal{A}$ if for any $w \in S$ the graph $\mathcal{E}(w)$ is a tree compatible with $<_L$ and $<_R$.

Planar dendric sets

Example

The *Tribonacci set* is **not** a planar dendric set.
Indeed, let us consider the extension graphs of the bispecial words ε, a and aba.



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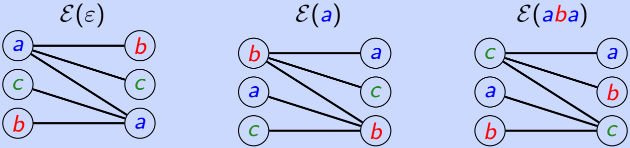
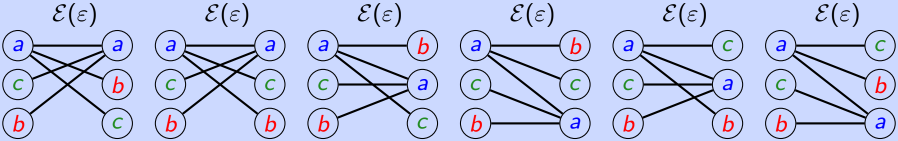


- $a <_L c <_L b$



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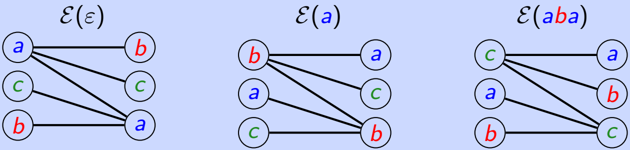
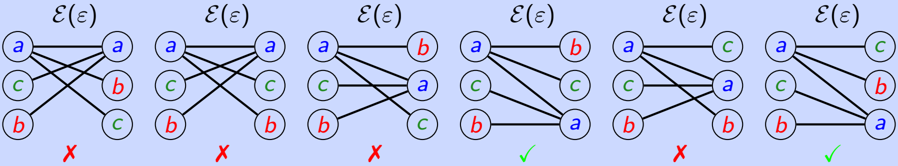


- $a <_L c <_L b \quad \Longrightarrow \quad b <_R c <_R a$ or $c <_R b <_R a$



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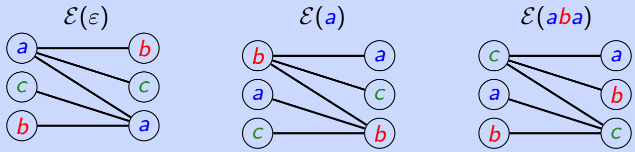

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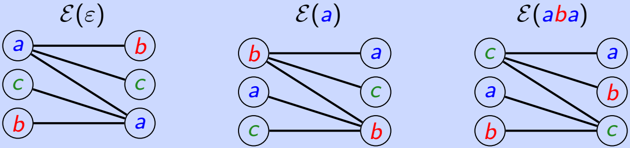

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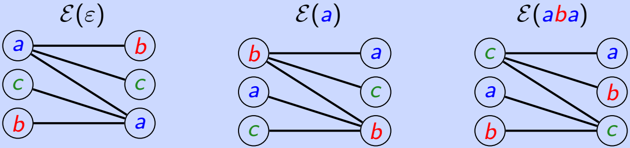

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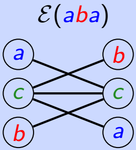

Planar dendric sets

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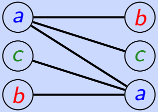
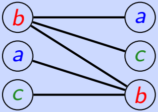
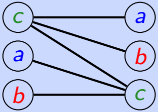
$\mathcal{E}(\varepsilon)$



$\mathcal{E}(a)$

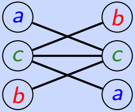


$\mathcal{E}(aba)$



- $a <_L c <_L b$ $\Longrightarrow$ ↯

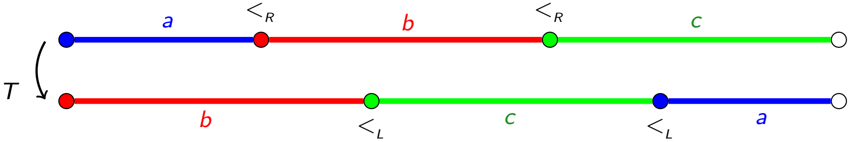
$\mathcal{E}(aba)$

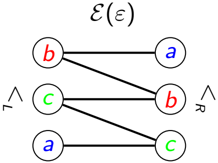


Planar dendric sets

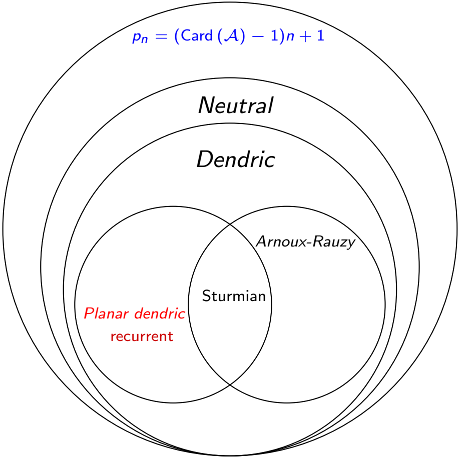
Theorem [Ferenczi, Zamboni (2008)]

A set S is a regular interval exchange set if and only if it is a *recurrent* planar dendric set.





Dendric and neutral sets



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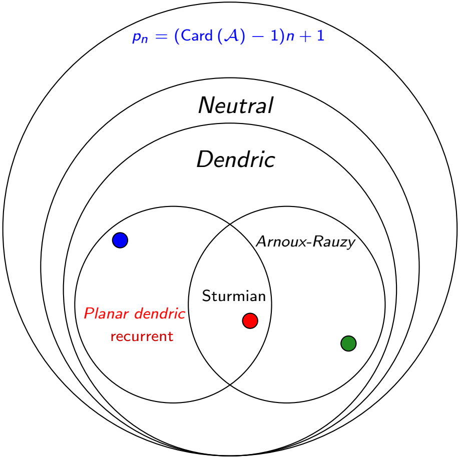

- Fibonacci
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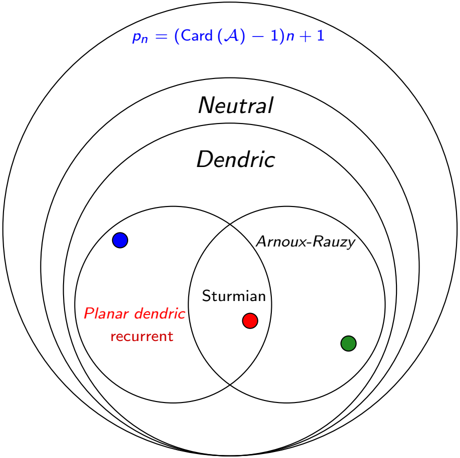

- • Fibonacci
- ? 2-coded Fibonacci
- • Tribonacci
- ? 2-coded Tribonacci
- • regular IE
- ? 2-coded regular IE

Bifix codes

Definition

A *bifix code* is a set $B \subset \mathcal{A}^+$ of nonempty words that does not contain any proper prefix or suffix of its elements.

Example

- ✓ $\{aa, ab, ba\}$
- ✓ $\{aa, ab, bba, bbb\}$
- ✓ $\{ac, bcc, bcbca\}$
- ✗ { choc, chocolate, vanille }
- ✗ { arbre, feuille, marbre }
- ✗ { aise, fraise, frai }

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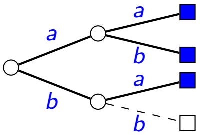
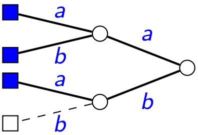
A bifix code $B \subset S$ is S-*maximal* if it is not properly contained in a bifix code $C \subset S$.

Example (Fibonacci)

The set $B = \{aa, ab, ba\}$ is an S-maximal bifix code.
It is not an $\mathcal{A}^*$-maximal bifix code, since $B \subset B \cup \{bb\}$.





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A *coding morphism* for a bifix code $B \subset A^+$ is a morphism $f : \mathcal{B}^* \to \mathcal{A}^*$ which maps bijectively $\mathcal{B}$ onto B.

Example

The map $f : \{u, v, w\}^* \to \{a, b\}^*$ is a coding morphism for $B = \{aa, ab, ba\}$.

$$f : \begin{cases} u \mapsto aa \\ v \mapsto ab \\ w \mapsto ba \end{cases}$$

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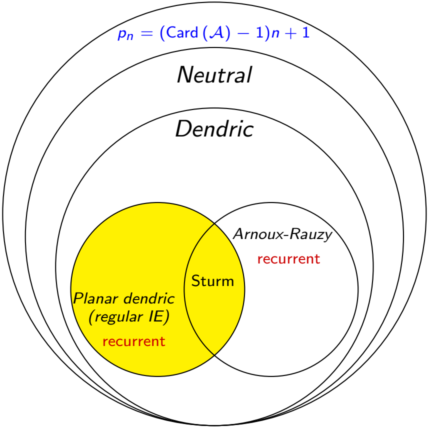
When S is factorial and B is an S-maximal bifix code, the set $f^{-1}(S)$ is called a *maximal bifix decoding* of S.

Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014)]

The family of *recurrent* planar dendric sets (i.e. regular interval exchange sets) is closed under maximal bifix decoding.



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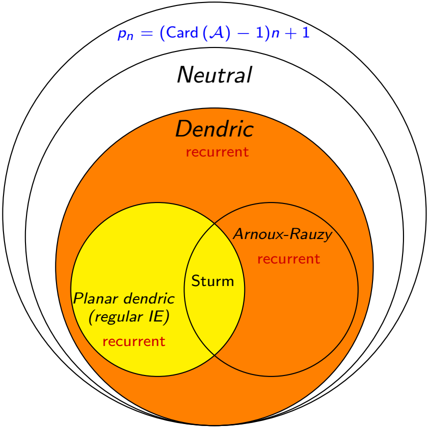
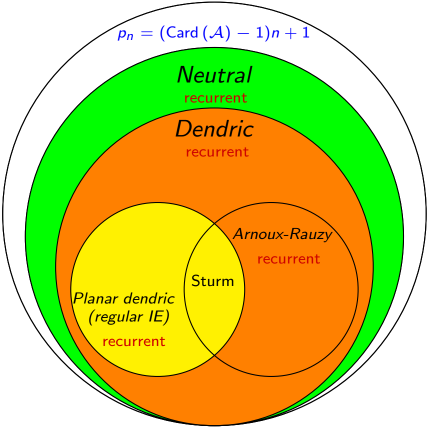


Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015) ; D., Perrin (2016)]

The family of *recurrent* neutral sets is closed under maximal bifix decoding.



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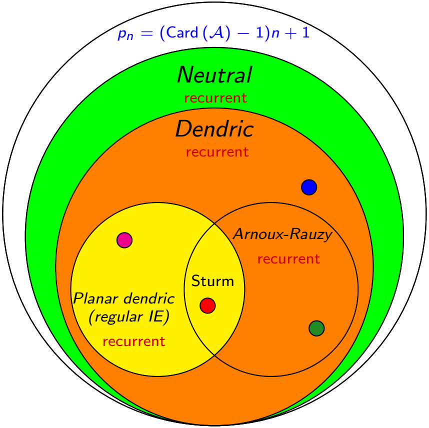

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A question by Fabien Durand

$$\mathbf{x} = abaababaabaababa \cdots$$

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$$\mathbf{x} = \boxed{ab}\,aababaabaababa \cdots$$

$$\sigma(\mathbf{x}) = v$$

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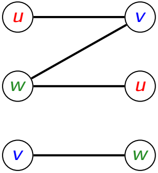

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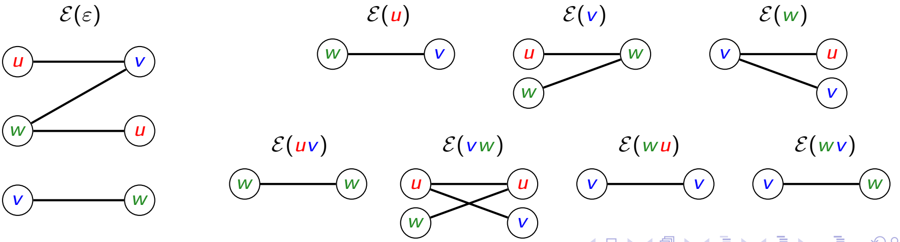

Eventually dendric sets

Definition

A biextendable factorial set S is called a *eventually dendric set* with threshold $m \geq 0$ if the graph $\mathcal{E}(w)$ is a tree for any $w \in S$ s.t. $|w| \geq m$.

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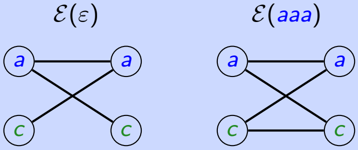
Example (coding of Tribonacci)

Let us consider the set $\alpha(S)$, where $\alpha : a, b \mapsto a, \quad c \mapsto c$.



The extension graph of all words of length at least 4 is a tree. (Just trust me !)

Eventually dendric sets

Complexity

Let us consider the function $s_n = p_{n+1} - p_n$.

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Proposition [D., Perrin (2019)]

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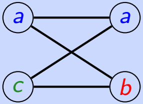
Example (the converse is not true)

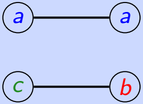
The *Chacon ternary set* is the set of factors of $\varphi^\omega(a)$, where $\varphi : \begin{cases} a \mapsto aabc \\ b \mapsto bc \\ c \mapsto abc \end{cases}$.

One has $p_n = 2n + 1$ ($\Rightarrow s_n = 2$).

$\mathcal{E}(abc)$



$\mathcal{E}(bca)$

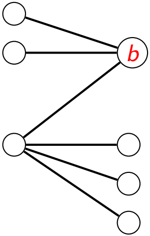


Eventually dendric sets

Theorem [D., Perrin (2019)]

A biextendable factorial set S is eventually dendric if and only if there exists $N \geq 0$ s.t. any left-special word $w \in S$ of length at least N has exactly one right extension wb that is left-special.

$\mathcal{E}(w)$



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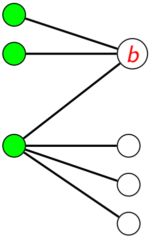
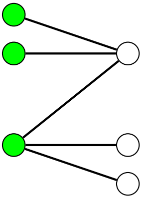
Moreover, in that case one has $L(wb) = L(w)$.

$\mathcal{E}(w)$



$\mathcal{E}(wb)$



Recurrence and uniformly recurrence

Definition

A factorial set S is *recurrent* if for every $u, v \in S$ there is a $w \in S$ such that uwv is in S.

Example (Fibonacci)

$$x = abaababaabaababaababaabaababaaba\cdots$$

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$$x = \underbrace{abaa}_{4}\,ba\,\underbrace{baab}_{4}\,\underbrace{aaba}_{4}\,baababaaba\,\underbrace{abab}_{4}\,a\cdots$$

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Proposition

Uniform recurrence $\Longrightarrow$ recurrence.

Return words

A (*right*) *return word* to w in S is a nonempty word u such that $wu \in S$ starts and ends with w but has no w as an internal factor. Formally,

$$\mathcal{R}_S(w) = \{u \in A^+ \mid wu \in S \cap (A^+w \setminus A^+wA^+)\}$$

Example (Fibonacci)

$\mathcal{R}_S(b) = \{a\underline{b}, aa\underline{b}\}$

$$\varphi(a)^\omega = abaaba\underline{b}aabaababaababaa\underline{b}aababaabaab\cdots$$

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$\mathcal{R}_S(aa) = \{b\underline{aa}, bab\underline{aa}\}$

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Cardinality of return words

Theorem [Vuillon (2001)]

Let S be a Sturmian set. For every $w \in S$, one has

$$\mathrm{Card}\,(\mathcal{R}_S(w)) = 2.$$

Cardinality of return words

Theorem [Vuillon (2001) ; Balková, Pelantová, Steiner (2008)]

Let S be a recurrent neutral set. For every $w \in S$, one has

$$\text{Card}\left(\mathcal{R}_S(w)\right) = \text{Card}\left(A\right).$$

Cardinality of return words

Theorem [Vuillon (2001) ; Balková, Pelantová, Steiner (2008) ; D., Perrin (2019)]

Let S be a recurrent eventually neutral set with threshold m. For every $w \in S$ with $|w| \geq$ m, one has

$$\mathrm{Card}\,(\mathcal{R}_S(w)) = 1 + \sum_{|u|<m} (\mathrm{Card}\,(R(u)) - 1).$$

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Corollary

An eventually neutral (dendric) set is recurrent if and only if it is uniformly recurrent

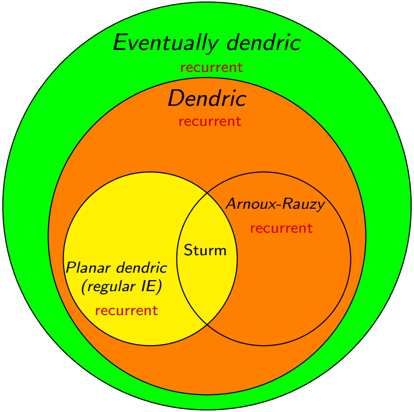
Proof. A recurrent set S is uniformly recurrent if and only if $\mathcal{R}_S(w)$ is finite for all $w \in S$.

Eventually dendric sets

Maximal bifix decoding

Theorem [D., Perrin (2019)]

The family of recurrent eventually dendric sets (with threshold m) is closed under maximal bifix decoding.



Dendric shift spaces

The *shift transformation* is the function

$$\begin{array}{rccc} \sigma : & \mathcal{A}^{\mathbb{Z}} & \rightarrow & \mathcal{A}^{\mathbb{Z}} \\ & (x_n)_{n\in\mathbb{Z}} & \mapsto & (x_{n+1})_{n\in\mathbb{Z}} \end{array}$$

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The pair (X, σ), with X a closed σ-invariant subset of $\mathcal{A}^{\mathbb{Z}}$ is called a *shift space*.

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(X, σ) is an *(eventually) dendric shift space* if its language $\mathcal{L}(X) = \bigcup_{\mathbf{x}\in X} \mathrm{Fac}\,(\mathbf{x})$ is an (eventually) dendric set.

Conjugacy

A map $\phi : X \to Y$ is called a *sliding block code* if it is continuous and $\phi \circ \sigma_X = \sigma_Y \circ \phi$.

$$\begin{array}{ccc} & \sigma_X & \\ X & \longrightarrow & X \\ \phi\downarrow & & \downarrow\phi \\ Y & \longrightarrow & Y \\ & \sigma_Y & \end{array}$$

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▷ *k-th higher block codes*, i.e., $\gamma_k : \begin{array}{c} X \longrightarrow X^{(k)} \subset \mathcal{A}_k^{\mathbb{Z}} \\ (x_n)_n \mapsto (y_n)_n \end{array} \quad y_n = f(x_n \cdots x_{n+k-1})$

Example (Fibonacci)

$$\gamma_k : \begin{array}{c} X \longrightarrow X^{(2)} \\ (x_n) \mapsto (y_n) \end{array} \qquad f : \begin{cases} \underline{a}a \mapsto u \\ \underline{a}b \mapsto v \\ \underline{b}a \mapsto w \end{cases}$$

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- *alphabetic morphisms*, i.e., $\alpha : \mathcal{A}^* \to \mathcal{B}^*$ s.t. $\alpha(\mathcal{A}) \subset \mathcal{B}$.

Example (Tribonacci, $\mathcal{A} = \{a, b, c\}$, $\mathcal{B} = \{a, c\}$)

$$\gamma_k : \begin{array}{c} X \longrightarrow Y \\ (x_n) \mapsto (y_n) \end{array} \quad \begin{cases} a \mapsto a \\ b \mapsto a \\ a \mapsto c \end{cases}$$

Conjugacy

Theorem [D., Perrin (2019)]

The family of eventually dendric sets is closed under conjugacy.

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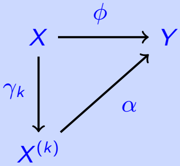
Proof.



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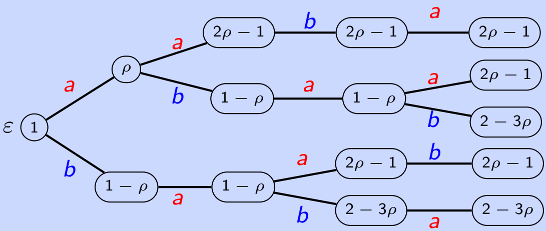
Theorem [Arnoux, Rauzy (1991)]

Shift spaces associated to Arnoux-Rauzy sets are uniquely ergodic.

Example (Fibonacci, $\rho = (\sqrt{5} - 1)/2$)

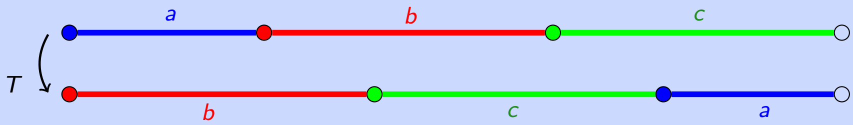


Ergodicity of dendric shift spaces

Given an interval exchange transformation T and a word $w = a_0 a_1 \cdots a_{m-1} \in \mathcal{A}^*$, let

$$I_w = I_{a_0} \cap T^{-1}(I_{a_1}) \cap \ldots \cap T^{-m+1}(I_{a_{m-1}})$$

Example



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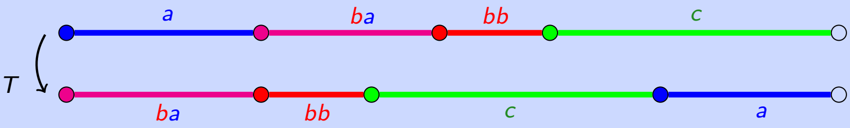

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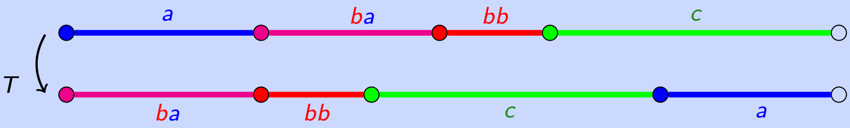

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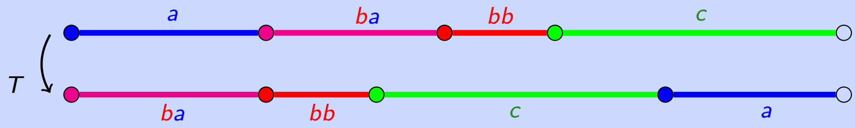

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QUESTION : Is it the only one ?

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Conjecture [Keane (1975)]

Every regular IE is uniquely ergodic.

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There exist regular IE not uniquely ergodic.

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Corollary

Dendric shift spaces are **not** in general uniquely ergodic (even when minimal).

Ergodicity of dendric shift spaces

Theorem [Boshernitzan (1984)]

A minimal symbolic system such that $\limsup_{n \to \infty} \left(\frac{p_n}{n} \right) < 3$ is uniquely ergodic.

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Minimal dendric shift spaces over an alphabet of size ≤ 3 are uniquely ergodic.


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Theorem [Damron, Fickenscher (2019)]

A minimal eventually dendric shift space has at most $\frac{\sup_n(s_n)+1}{2}$ ergodic measurs.

Minimal dendric shift spaces on a 3-letter alphabet

Two shift spaces $(X, \sigma), (Y, \sigma)$ are *orbit equivalent* if there exists a homeomorphism $\eta : X \to Y$ such that for all $\mathbf{x} \in X$ one has

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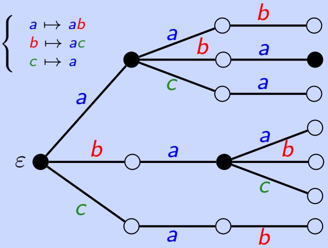
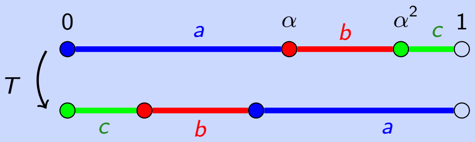
Theorem [Berthé, Cecchi, D., Durand, Leroy, Perrin, Petite (2018+)]

All minimal dendric shift spaces on a 3 letter alphabet having the same letter frequency are orbit equivalent.

Example $(\alpha + \alpha^2 + \alpha^3 = 1)$

$$\psi : \begin{cases} a \mapsto ab \\ b \mapsto ac \\ c \mapsto a \end{cases}$$





Open questions

- Subgroup generated by $\mathcal{R}(w)$ in an eventually dendric set ?

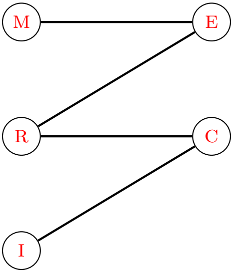
  [For a dendric set, $\mathcal{R}(w)$ is a basis of the free group on $\mathcal{A}$.]

- Closure under taking factors ?

  [Y is a *factor* of X, if there is a sliding block (not necessary bijective) $\phi : X \to Y$]

- Decidability of the (eventually) dendric condition.

  [Work in progress with Revekka Kyriakoglou and Julien Leroy]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