Return words and bifix codes in eventually dendric sets

Francesco Dolce and Dominique Perrin*

*Université Paris-Est Marne-la-Vallée (France)

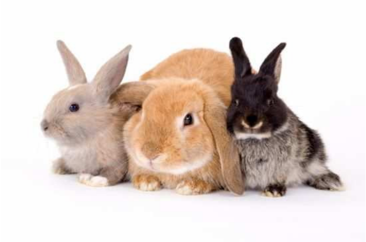
WORDS 2019

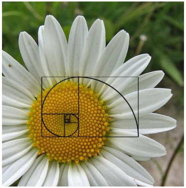
Loughborough, 12 September 2019

Fibonacci

$$\mathbf{x} = abaababaabaababa\cdots$$

$$\mathbf{x} = \lim_{n\to\infty} \varphi^n(a) \qquad \text{where} \qquad \varphi : \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$





Fibonacci

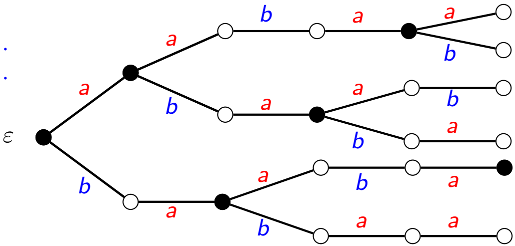
$$\mathbf{x} = abaababaabaababa \cdots$$

The *Fibonacci set* (set of factors of $\mathbf{x}$) is a Sturmian set.

Definition

A *Sturmian* set $S \subset \mathcal{A}^*$ is a factorial set such that $p_n = \mathrm{Card}\,(S \cap \mathcal{A}^n) = n + 1$.

n :	0	1	2	3	4	5	$\cdots$
p_n :	1	2	3	4	5	6	$\cdots$



2-coded Fibonacci

$$x = ab\ aa\ ba\ ba\ ab\ aa\ ba\ ba\ \cdots$$

2-coded Fibonacci

$$x = ab\ aa\ ba\ ba\ ab\ aa\ ba\ ba\ \cdots$$

$$f : \begin{cases} u & \mapsto & aa \\ v & \mapsto & ab \\ w & \mapsto & ba \end{cases}$$

2-coded Fibonacci

$$x = ab\ aa\ ba\ ba\ ab\ aa\ ba\ ba\ \cdots$$

$$f^{-1}(x) = v\ u\ w\ w\ v\ u\ w\ w\ \cdots$$

$$f : \begin{cases} u & \mapsto & aa \\ v & \mapsto & ab \\ w & \mapsto & ba \end{cases}$$

2-coded Fibonacci

$$x = ab\ aa\ ba\ ba\ ab\ aa\ ba\ ba\ \cdots$$

$$f^{-1}(x) = v\ u\ w\ w\ v\ u\ w\ w\ \cdots$$

$$f : \begin{cases} u \mapsto aa \\ v \mapsto ab \\ w \mapsto ba \end{cases}$$

Arnoux-Rauzy sets

Definition

An *Arnoux-Rauzy* set is a factorial set closed by reversal with $p_n = (\mathrm{Card}\,(\mathcal{A}) - 1)n + 1$ having a unique right special factor for each length.

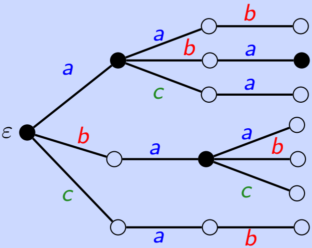
Arnoux-Rauzy sets

Definition

An *Arnoux-Rauzy* set is a factorial set closed by reversal with $p_n = (\text{Card}\,(\mathcal{A}) - 1)n + 1$ having a unique right special factor for each length.

Example (Tribonacci)

Factors of the fixed point $\psi^\omega(a)$ of the morphism $\psi : a \mapsto ab, \quad b \mapsto ac, \quad c \mapsto a.$



$n:$	0	1	2	3	$\cdots$
$p_n:$	1	3	5	7	$\cdots$

$$p_n = 2n + 1$$

2-coded Fibonacci

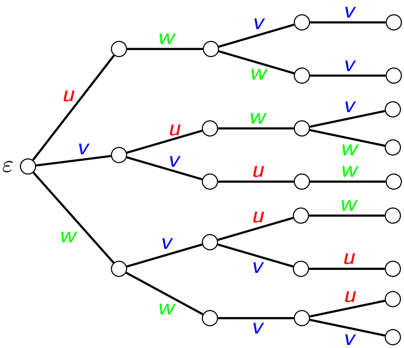
$$f^{-1}(x) = v\ u\ w\ w\ v\ u\ w\ w\ \cdots$$

Is the set of factors of $f^{-1}(x)$ an Arnoux-Rauzy set?

2-coded Fibonacci

$$f^{-1}(x) = v\ u\ w\ w\ v\ u\ w\ w\ \cdots$$

Is the set of factors of $f^{-1}(x)$ an Arnoux-Rauzy set?



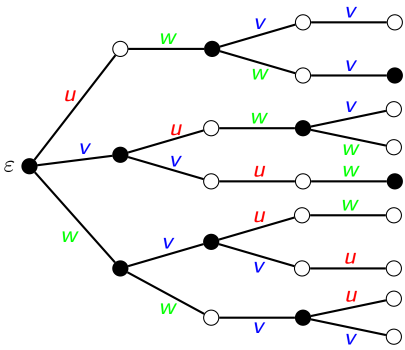
$$p_n = 2n + 1$$

n :	0	1	2	3	4	$\cdots$
p_n :	1	3	5	7	9	$\cdots$

2-coded Fibonacci

$$f^{-1}(x) = v\ u\ w\ w\ v\ u\ w\ w\ \cdots$$

Is the set of factors of $f^{-1}(x)$ an Arnoux-Rauzy set? No!



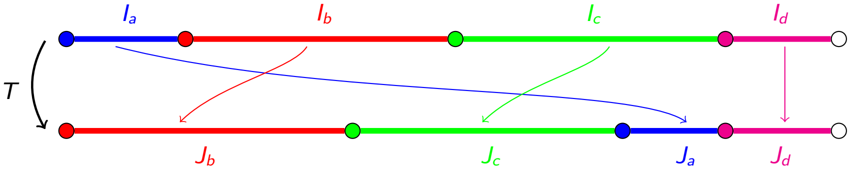
$$p_n = 2n + 1$$

n :	0	1	2	3	4	$\cdots$
p_n :	1	3	5	7	9	$\cdots$

Interval exchanges

Let $(I_\alpha)_{\alpha\in\mathcal{A}}$ and $(J_\alpha)_{\alpha\in\mathcal{A}}$ be two partitions of $[0,1[$.
An *interval exchange transformation* (IET) is a map $T : [0,1[\to [0,1[$ defined by

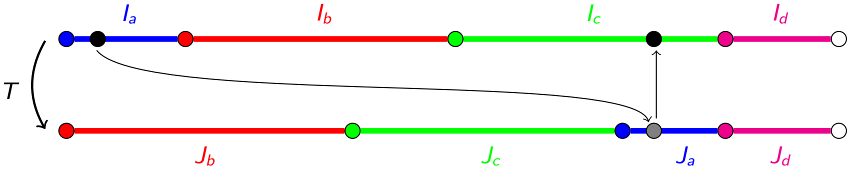
$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha.$$



Interval exchanges

Let $(I_\alpha)_{\alpha\in\mathcal{A}}$ and $(J_\alpha)_{\alpha\in\mathcal{A}}$ be two partitions of $[0,1[$.
An *interval exchange transformation* (IET) is a map $T : [0,1[\to [0,1[$ defined by

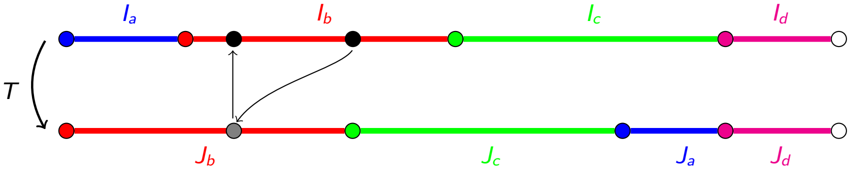
$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha.$$



Interval exchanges

Let $(I_\alpha)_{\alpha\in\mathcal{A}}$ and $(J_\alpha)_{\alpha\in\mathcal{A}}$ be two partitions of $[0,1[$.
An *interval exchange transformation* (IET) is a map $T : [0,1[\to [0,1[$ defined by

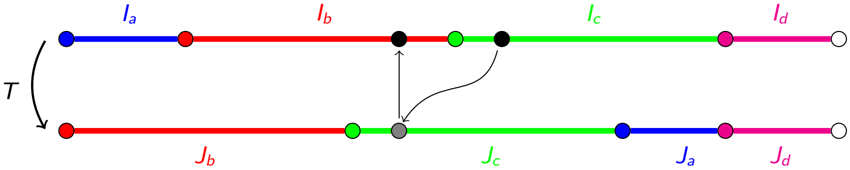
$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha.$$



Interval exchanges

Let $(I_\alpha)_{\alpha\in\mathcal{A}}$ and $(J_\alpha)_{\alpha\in\mathcal{A}}$ be two partitions of $[0,1[$.
An *interval exchange transformation* (IET) is a map $T : [0,1[\to [0,1[$ defined by

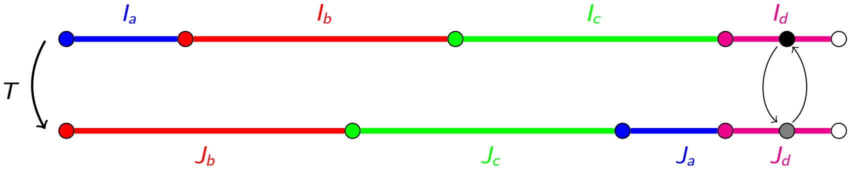
$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha .$$



Interval exchanges

Let $(I_\alpha)_{\alpha\in\mathcal{A}}$ and $(J_\alpha)_{\alpha\in\mathcal{A}}$ be two partitions of $[0,1[$.
An *interval exchange transformation* (IET) is a map $T : [0,1[\to [0,1[$ defined by

$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha.$$



Interval exchanges

T is said to be *minimal* if for any point $z \in [0, 1[$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[0, 1[$.

T is said *regular* if the orbits of the non-zero separation points are infinite and disjoint.

Theorem [Keane (1975)]

A regular interval exchange transformation is minimal.

Interval exchanges

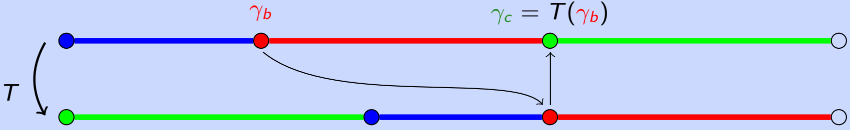
T is said to be *minimal* if for any point $z \in [0,1[$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[0,1[$.

T is said *regular* if the orbits of the non-zero separation points are infinite and disjoint.

Theorem [Keane (1975)]

A regular interval exchange transformation is minimal.

Example (the converse is not true)

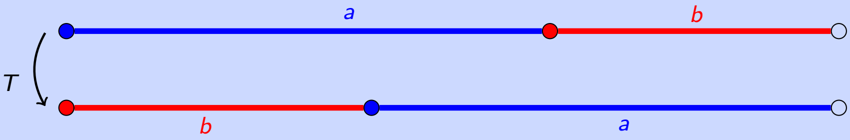


Interval exchanges

The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in \mathcal{A}^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha .$$

Example (Fibonacci, $z = (3 - \sqrt{5})/2$)

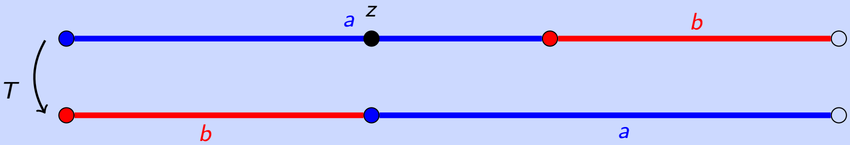


Interval exchanges

The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in \mathcal{A}^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha .$$

Example (Fibonacci, $z = (3-\sqrt{5})/2$)



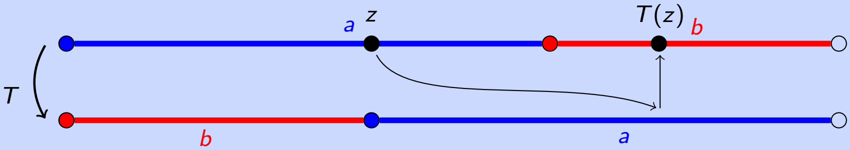
$$\Sigma_T(z) = a$$

Interval exchanges

The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in \mathcal{A}^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha.$$

Example (Fibonacci, $z = (3-\sqrt{5})/2$)



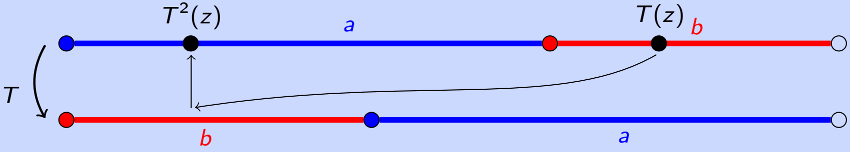
$$\Sigma_T(z) = ab$$

Interval exchanges

The *natural coding* of T relative to $z \in [0, 1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in \mathcal{A}^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha.$$

Example (Fibonacci, $z = (3 - \sqrt{5})/2$)



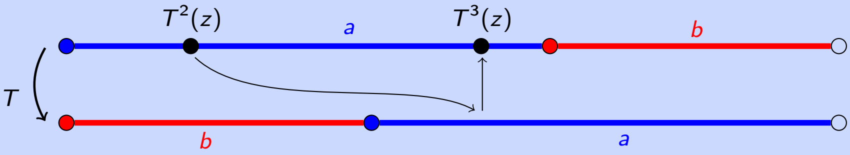
$$\Sigma_T(z) = aba$$

Interval exchanges

The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in \mathcal{A}^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha .$$

Example (Fibonacci, $z = (3-\sqrt{5})/2$)



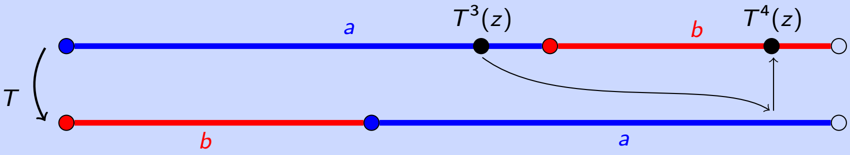
$$\Sigma_T(z) = abaa$$

Interval exchanges

The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in \mathcal{A}^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha.$$

Example (Fibonacci, $z = (3-\sqrt{5})/2$)



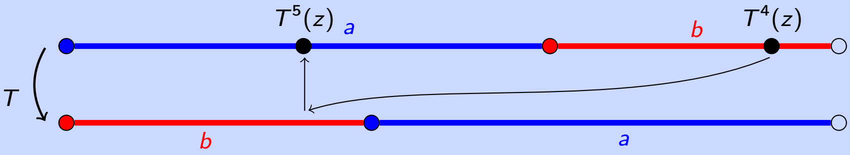
$$\Sigma_T(z) = abaab$$

Interval exchanges

The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in \mathcal{A}^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha .$$

Example (Fibonacci, $z = (3-\sqrt{5})/2$)



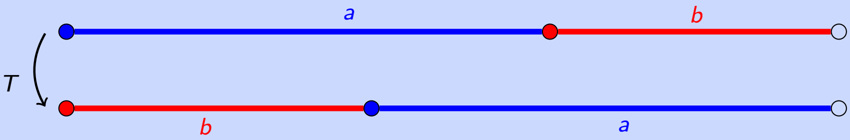
$$\Sigma_T(z) = abaaba \cdots$$

Interval exchanges

The set $\mathcal{L}(T) = \bigcup_{z \in [0,1[} \mathrm{Fac}\,(\Sigma_T(z))$ is said a (*minimal*, *regular*) *interval exchange set*.

Remark. If T is minimal, $\mathrm{Fac}\,(\Sigma_T(z))$ does not depend on the point z.

Example (Fibonacci)



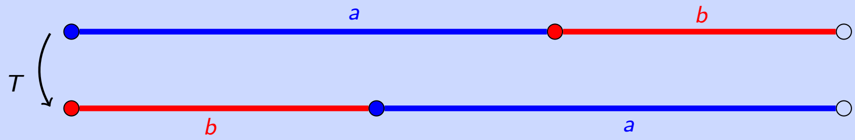
$$\mathcal{L}(T) = \left\{ \varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab\ \ldots \right\}$$

Interval exchanges

The set $\mathcal{L}(T) = \bigcup_{z \in [0,1[} \mathrm{Fac}\,(\Sigma_T(z))$ is said a (*minimal*, *regular*) *interval exchange set*.

Remark. If T is minimal, $\mathrm{Fac}\,(\Sigma_T(z))$ does not depend on the point z.

Example (Fibonacci)

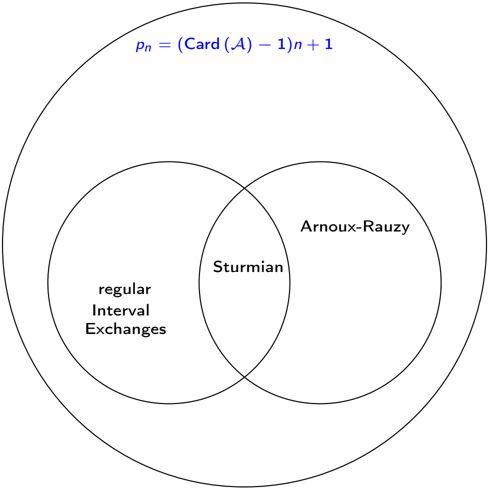


$$\mathcal{L}(T) = \{\varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab\ \ldots\}$$

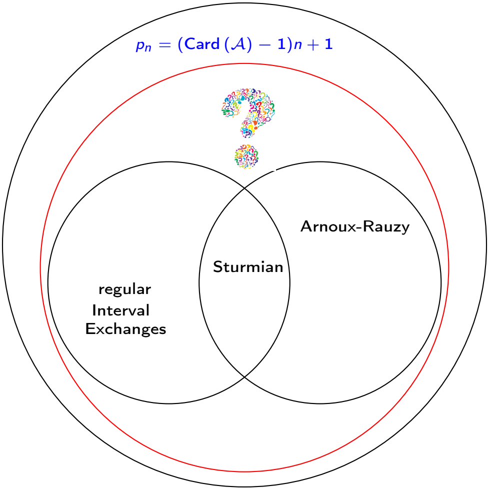
Proposition

Regular interval exchange sets have factor complexity $p_n = (\mathrm{Card}\,(\mathcal{A}) - 1)n + 1$.

Arnoux-Rauzy and Interval exchanges



Arnoux-Rauzy and Interval exchanges

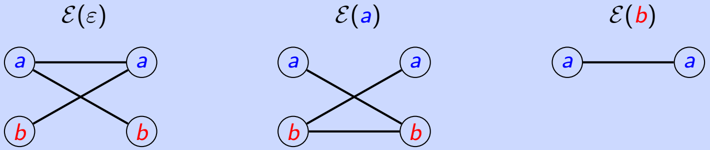


Extension graphs

The *extension graph* of a word $w \in \mathcal{L}$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{u \in \mathcal{A} \mid uw \in \mathcal{L}\} \\ R(w) &= \{v \in \mathcal{A} \mid wv \in \mathcal{L}\} \\ B(w) &= \{(u, v) \in \mathcal{A} \times \mathcal{A} \mid uwv \in \mathcal{L}\} \end{aligned}$$

Example (Fibonacci, $\mathcal{L} = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, bab, \ldots\}$)

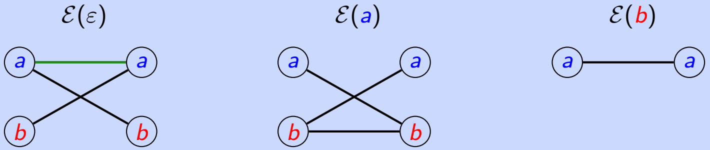


Extension graphs

The *extension graph* of a word $w \in \mathcal{L}$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{u \in \mathcal{A} \mid uw \in \mathcal{L}\} \\ R(w) &= \{v \in \mathcal{A} \mid wv \in \mathcal{L}\} \\ B(w) &= \{(u, v) \in \mathcal{A} \times \mathcal{A} \mid uwv \in \mathcal{L}\} \end{aligned}$$

Example (Fibonacci, $\mathcal{L} = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, bab, \ldots\}$)

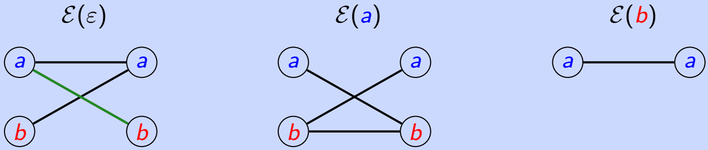


Extension graphs

The *extension graph* of a word $w \in \mathcal{L}$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{u \in \mathcal{A} \mid uw \in \mathcal{L}\} \\ R(w) &= \{v \in \mathcal{A} \mid wv \in \mathcal{L}\} \\ B(w) &= \{(u,v) \in \mathcal{A} \times \mathcal{A} \mid uwv \in \mathcal{L}\} \end{aligned}$$

Example (Fibonacci, $\mathcal{L} = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, bab, \ldots\}$)

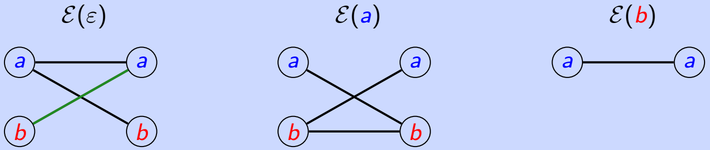


Extension graphs

The *extension graph* of a word $w \in \mathcal{L}$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{u \in \mathcal{A} \mid uw \in \mathcal{L}\} \\ R(w) &= \{v \in \mathcal{A} \mid wv \in \mathcal{L}\} \\ B(w) &= \{(u,v) \in \mathcal{A} \times \mathcal{A} \mid uwv \in \mathcal{L}\} \end{aligned}$$

Example (Fibonacci, $\mathcal{L} = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, bab, \ldots\}$)

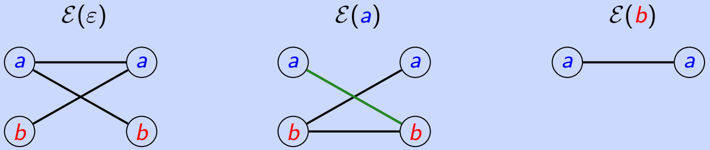


Extension graphs

The *extension graph* of a word $w \in \mathcal{L}$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{u \in \mathcal{A} \mid uw \in \mathcal{L}\} \\ R(w) &= \{v \in \mathcal{A} \mid wv \in \mathcal{L}\} \\ B(w) &= \{(u, v) \in \mathcal{A} \times \mathcal{A} \mid uwv \in \mathcal{L}\} \end{aligned}$$

Example (Fibonacci, $\mathcal{L} = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, bab, \ldots\}$)



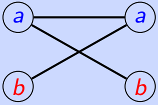
Extension graphs

The *extension graph* of a word $w \in \mathcal{L}$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

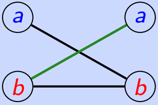
$$
\begin{aligned}
L(w) &= \{u \in \mathcal{A} \mid uw \in \mathcal{L}\} \\
R(w) &= \{v \in \mathcal{A} \mid wv \in \mathcal{L}\} \\
B(w) &= \{(u, v) \in \mathcal{A} \times \mathcal{A} \mid uwv \in \mathcal{L}\}
\end{aligned}
$$

Example (Fibonacci, $\mathcal{L} = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, bab, \ldots\}$)

$\mathcal{E}(\varepsilon)$



$\mathcal{E}(a)$



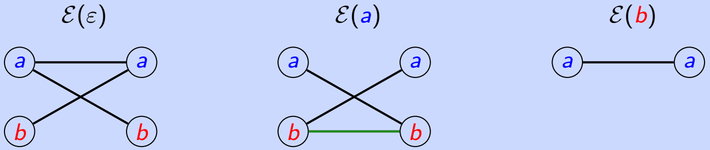
$\mathcal{E}(b)$

Extension graphs

The *extension graph* of a word $w \in \mathcal{L}$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{u \in \mathcal{A} \mid uw \in \mathcal{L}\} \\ R(w) &= \{v \in \mathcal{A} \mid wv \in \mathcal{L}\} \\ B(w) &= \{(u,v) \in \mathcal{A} \times \mathcal{A} \mid uwv \in \mathcal{L}\} \end{aligned}$$

Example (Fibonacci, $\mathcal{L} = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, bab, \ldots\}$)

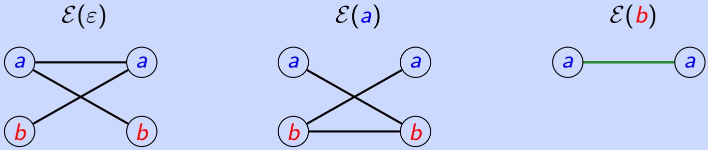


Extension graphs

The *extension graph* of a word $w \in \mathcal{L}$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{u \in \mathcal{A} \mid uw \in \mathcal{L}\} \\ R(w) &= \{v \in \mathcal{A} \mid wv \in \mathcal{L}\} \\ B(w) &= \{(u,v) \in \mathcal{A} \times \mathcal{A} \mid uwv \in \mathcal{L}\} \end{aligned}$$

Example (Fibonacci, $\mathcal{L} = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, bab, \ldots\}$)



Extension graphs

The *extension graph* of a word $w \in \mathcal{L}$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

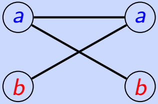
$$\begin{aligned} L(w) &= \{u \in \mathcal{A} \mid uw \in \mathcal{L}\} \\ R(w) &= \{v \in \mathcal{A} \mid wv \in \mathcal{L}\} \\ B(w) &= \{(u,v) \in \mathcal{A} \times \mathcal{A} \mid uwv \in \mathcal{L}\} \end{aligned}$$

The *multiplicity* of a word w is the quantity

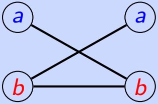
$$m(w) = \mathrm{Card}\,(B(w)) - \mathrm{Card}\,(L(w)) - \mathrm{Card}\,(R(w)) + 1.$$

Example (Fibonacci, $\mathcal{L} = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, bab, \ldots\}$)

$\mathcal{E}(\varepsilon)$



$\mathcal{E}(a)$



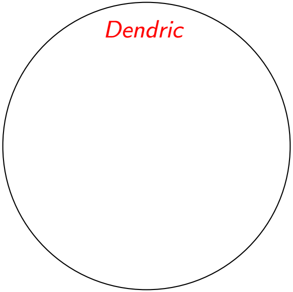
$\mathcal{E}(b)$

$m(a) = 3 - 2 - 2 + 1 = 0$

Dendric and neutral sets

Definition

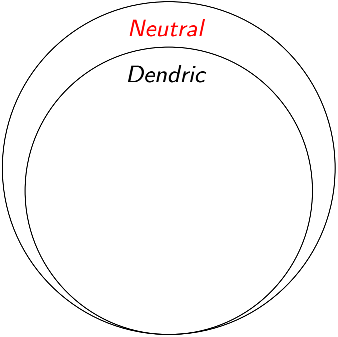
A language $\mathcal{L}$ is called (purely) *dendric* if the graph $\mathcal{E}(w)$ is a tree for any $w \in \mathcal{L}$.



Dendric and neutral sets

Definition

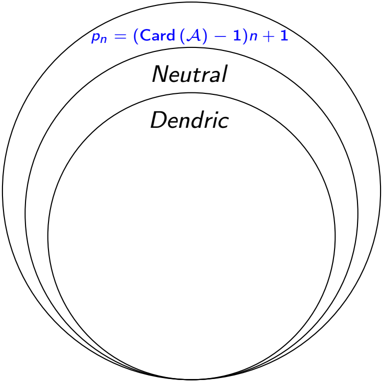
A language $\mathcal{L}$ is called (purely) *dendric* if the graph $\mathcal{E}(w)$ is a tree for any $w \in \mathcal{L}$. It is called *neutral* if every word w has multiplicity $m(w) = 0$.



Dendric and neutral sets

Definition

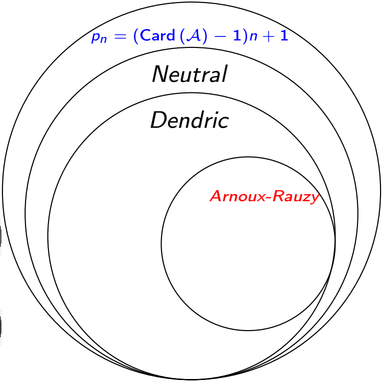
A language $\mathcal{L}$ is called (purely) *dendric* if the graph $\mathcal{E}(w)$ is a tree for any $w \in \mathcal{L}$. It is called *neutral* if every word w has multiplicity $m(w) = 0$.



Dendric and neutral sets

Definition

A language $\mathcal{L}$ is called (purely) *dendric* if the graph $\mathcal{E}(w)$ is a tree for any $w \in \mathcal{L}$. It is called *neutral* if every word w has multiplicity $m(w) = 0$.

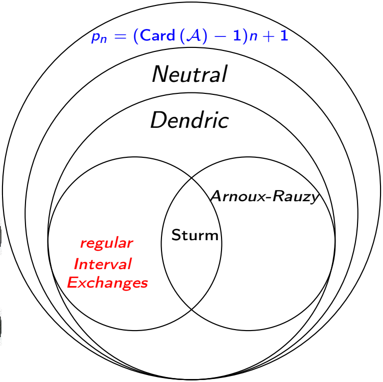


[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone: "**Acyclic, connected and tree sets**" (2014).]

Dendric and neutral sets

Definition

A language $\mathcal{L}$ is called (purely) *dendric* if the graph $\mathcal{E}(w)$ is a tree for any $w \in \mathcal{L}$. It is called *neutral* if every word w has multiplicity $m(w) = 0$.

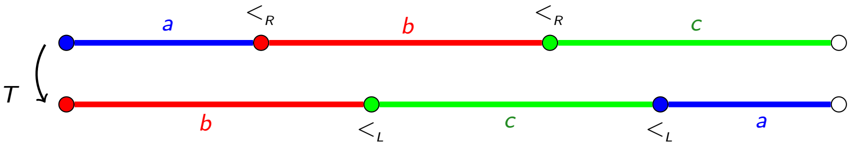


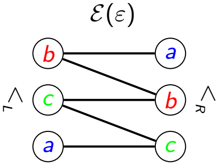
[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone: "Bifix codes and interval exchanges" (2015).]

Planar dendric sets

Theorem [S. Ferenczi, L. Zamboni (2008)]

A set S is a regular interval exchange set if and only if it is a recurrent *planar dendric set*.





Eventually dendric sets

Definition

A language $\mathcal{L}$ is called *eventually dendric* with threshold $m \geq 0$ if the graph $\mathcal{E}(w)$ is a tree for any $w \in \mathcal{L}^{\geq m}$.

Eventually dendric sets

Definition

A language $\mathcal{L}$ is called *eventually dendric* with threshold $m \geq 0$ if the graph $\mathcal{E}(w)$ is a tree for any $w \in \mathcal{L}^{\geq m}$.
It is called *eventually neutral* if every word $w \in \mathcal{L}^{\geq m}$ has multiplicity $m(w) = 0$.

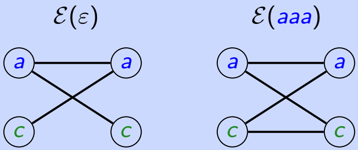
Eventually dendric sets

Definition

A language $\mathcal{L}$ is called *eventually dendric* with threshold $m \geq 0$ if the graph $\mathcal{E}(w)$ is a tree for any $w \in \mathcal{L}^{\geq m}$.
It is called *eventually neutral* if every word $w \in \mathcal{L}^{\geq m}$ has multiplicity $m(w) = 0$.

Example (coding of Tribonacci)

Let us consider the set $\alpha(S)$, where $\alpha : a, b \mapsto a, \quad c \mapsto c$.



The extension graph of all words of length at least 4 is a tree. (Just trust me!)

Eventually dendric sets

Complexity

Let us consider the function $s_n = p_{n+1} - p_n$.

Eventually dendric sets

Complexity

Let us consider the function $s_n = p_{n+1} - p_n$.

Proposition [D., Perrin (2019)]

Let $\mathcal{L}$ be eventually dendric. Then s_n is eventually constant.

Eventually dendric sets

Complexity

Let us consider the function $s_n = p_{n+1} - p_n$.

Proposition [D., Perrin (2019)]

Let $\mathcal{L}$ be eventually dendric. Then s_n is eventually constant.

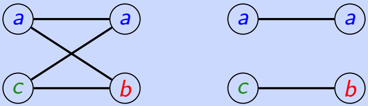
Example (the converse is not true)

The *Chacon ternary set* is the language arising from the morphism

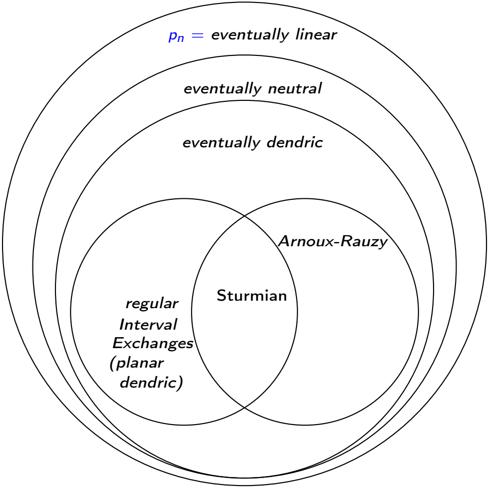
$$\varphi : a \mapsto aabc, \quad b \mapsto bc, \quad c \mapsto abc.$$

One has $p_n = 2n + 1$ ($\Rightarrow s_n = 2$). BUT for infinitly many pairs of words:

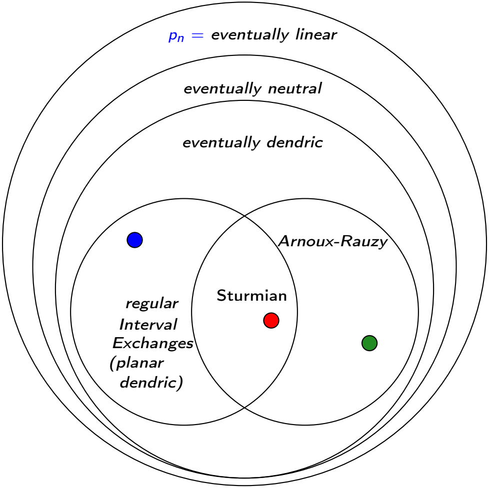
$\mathcal{E}(abc)$ $\qquad$ $\mathcal{E}(bca)$



Eventually dendric and eventually neutral sets

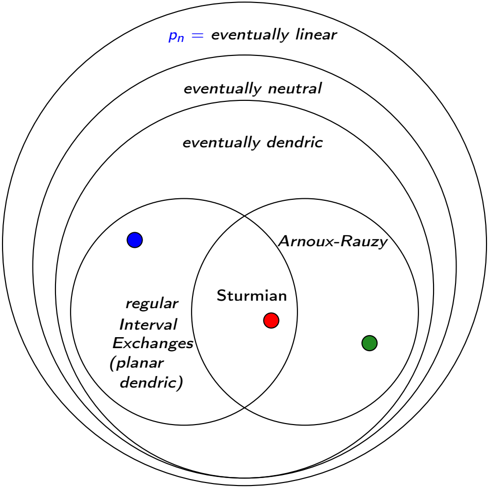


Eventually dendric and eventually neutral sets



- Fibonacci
- Tribonacci
- regular IE

Eventually dendric and eventually neutral sets



- Fibonacci
- ? 2-coded Fibonacci
- Tribonacci
- ? 2-coded Tribonacci
- regular IE
- ? 2-coded regular IE

Bifix codes

Definition

A *bifix code* is a set $B \subset \mathcal{A}^+$ of nonempty words that does not contain any proper prefix or suffix of its elements.

Example

- ✓ $\{aa, ab, ba\}$
- ✓ $\{aa, ab, bba, bbb\}$
- ✓ $\{ac, bcc, bcbca\}$
- ✗ { even, eventually, dendric }
- ✗ { borough, district, loughborough }
- ✗ { stone, stoneywell, well }

Bifix codes

Definition

A *bifix code* is a set $B \subset \mathcal{A}^+$ of nonempty words that does not contain any proper prefix or suffix of its elements.

A bifix code $B \subset S$ is *S-maximal* if it is not properly contained in a bifix code $C \subset S$.

Example (Fibonacci)

The set $B = \{aa, ab, ba\}$ is an S-maximal bifix code.
It is not an $\mathcal{A}^*$-maximal bifix code, since $B \subset B \cup \{bb\}$.

Bifix codes

Definition

A *bifix code* is a set $B \subset \mathcal{A}^+$ of nonempty words that does not contain any proper prefix or suffix of its elements.

A bifix code $B \subset S$ is *S-maximal* if it is not properly contained in a bifix code $C \subset S$.

A *coding morphism* for a bifix code $B \subset A^+$ is a morphism $f : \mathcal{B}^* \to \mathcal{A}^*$ which maps bijectively $\mathcal{B}$ onto B.

Example

The map $f : \{u, v, w\}^* \to \{a, b\}^* $ is a coding morphism for $B = \{aa, ab, ba\}$.

$$f : \begin{cases} u \mapsto aa \\ v \mapsto ab \\ w \mapsto ba \end{cases}$$

Bifix codes

Definition

A *bifix code* is a set $B \subset \mathcal{A}^+$ of nonempty words that does not contain any proper prefix or suffix of its elements.

A bifix code $B \subset S$ is *S-maximal* if it is not properly contained in a bifix code $C \subset S$.

A *coding morphism* for a bifix code $B \subset A^+$ is a morphism $f : \mathcal{B}^* \to \mathcal{A}^*$ which maps bijectively $\mathcal{B}$ onto B.

Example

The map $f : \{u, v, w\}^* \to \{a, b\}^*$ is a coding morphism for $B = \{aa, ab, ba\}$.

$$f : \begin{cases} u \mapsto aa \\ v \mapsto ab \\ w \mapsto ba \end{cases}$$

When S is factorial and B is an S-maximal bifix code, the set $f^{-1}(S)$ is called a *maximal bifix decoding* of S.

Recurrence and uniform recurrence

Definition

A language $\mathcal{L}$ is *recurrent* if for every $u, v \in \mathcal{L}$ there is a $w \in \mathcal{L}$ such that uwv is in $\mathcal{L}$.

Example (Fibonacci)

$$x = abaababaabaababaababaabaababaaba \cdots$$

Recurrence and uniform recurrence

Definition

A language $\mathcal{L}$ is *recurrent* if for every $u, v \in \mathcal{L}$ there is a $w \in \mathcal{L}$ such that uwv is in $\mathcal{L}$.

$\mathcal{L}$ is *uniformly recurrent* if for every $u \in S$ there exists an $n \in \mathbb{N}$ such that u is a factor of every word of length n in S.

Example (Fibonacci)

$$x = \underbrace{abaa}_{4}\, ba\, \underbrace{baab}_{4}\, \underbrace{aaba}_{4}\, baababaaba\, \underbrace{abab}_{4}\, a \cdots$$

Recurrence and uniform recurrence

Definition

A language $\mathcal{L}$ is *recurrent* if for every $u, v \in \mathcal{L}$ there is a $w \in \mathcal{L}$ such that uwv is in $\mathcal{L}$.

$\mathcal{L}$ is *uniformly recurrent* if for every $u \in S$ there exists an $n \in \mathbb{N}$ such that u is a factor of every word of length n in S.

- Arnoux-Rauzy
- regular Interval Exchanges

Recurrence and uniform recurrence

Definition

A language $\mathcal{L}$ is *recurrent* if for every $u, v \in \mathcal{L}$ there is a $w \in \mathcal{L}$ such that uwv is in $\mathcal{L}$.

$\mathcal{L}$ is *uniformly recurrent* if for every $u \in S$ there exists an $n \in \mathbb{N}$ such that u is a factor of every word of length n in S.

- Arnoux-Rauzy
- regular Interval Exchanges

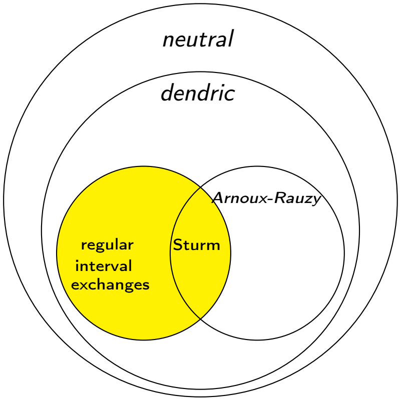
Proposition

Uniform recurrence $\Longrightarrow$ Recurrence.

Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014)]

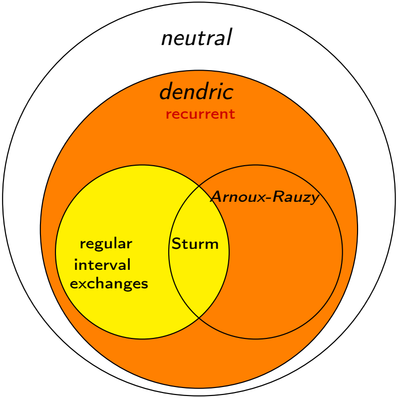
The family of regular interval exchanges sets is closed under maximal bifix decoding.



Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015)]

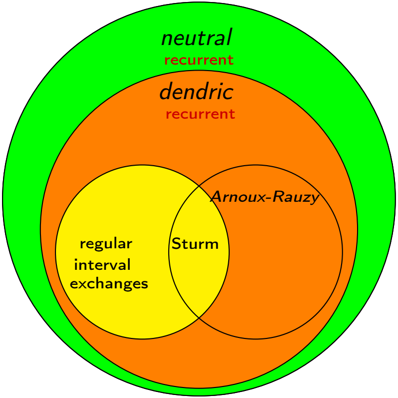
The family of *recurrent* dendric sets is closed under maximal bifix decoding.



Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015); D., Perrin (2016)]

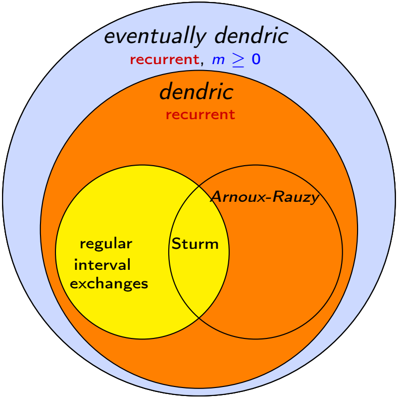
The family of *recurrent* neutral sets is closed under maximal bifix decoding.



Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015); D., Perrin (2016, 2019)]

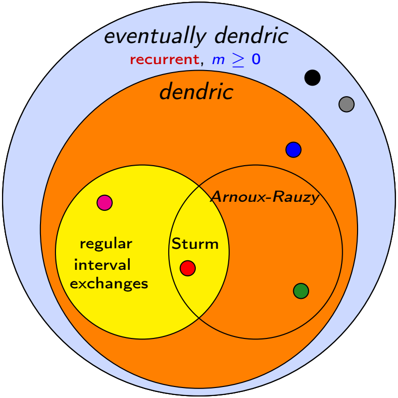
The family of *recurrent* eventually dendric sets of threshold m is closed under maximal bifix decoding.



Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015); D., Perrin (2016, 2019)]

The family of *recurrent* eventually dendric sets of threshold m is closed under maximal bifix decoding.



- Fibonacci
- 2-coded Fibonacci
- Tribonacci
- 2-coded Tribonacci
- α(Tribonacci)
- 2-coded α(Tribonacci)

Return words

A (*right*) *return word* to w in $\mathcal{L}$ is a nonempty word u such that $wu \in \mathcal{L}$ starts and ends with w but has no w as an internal factor. Formally,

$$\mathcal{R}(w) = \{u \in A^+ \mid wu \in \mathcal{L} \cap (A^+w \setminus A^+wA^+)\}$$

Example (Fibonacci)

$\mathcal{R}(b) = \{a\underline{b}, aa\underline{b}\}$

$$\varphi(a)^\omega = abaabab\underline{b}aabaababaabab\underline{b}aabaababaabaab\cdots$$

Return words

A (*right*) *return word* to w in $\mathcal{L}$ is a nonempty word u such that $wu \in \mathcal{L}$ starts and ends with w but has no w as an internal factor. Formally,

$$\mathcal{R}(w) = \{u \in A^+ \mid wu \in \mathcal{L} \cap (A^+ w \setminus A^+ w A^+)\}$$

Example (Fibonacci)

$\mathcal{R}(aa) = \{b\underline{aa}, bab\underline{aa}\}$

$$\varphi(a)^\omega = abaababa\underline{a}baababaababaabaababaabaab\cdots$$

Cardinality of return words

Theorem [Vuillon (2001)]

Let $\mathcal{L}$ be a Sturmian set. For every $w \in \mathcal{L}$, one has

$$\mathrm{Card}\,(\mathcal{R}(w)) = 2.$$

Cardinality of return words

Theorem [Vuillon (2001); Balková, Pelantová, Steiner (2008)]

Let $\mathcal{L}$ be a recurrent neutral set. For every $w \in \mathcal{L}$, one has

$$\mathrm{Card}\,(\mathcal{R}(w)) = \mathrm{Card}\,(A)\,.$$

Cardinality of return words

Theorem [Vuillon (2001); Balková, Pelantová, Steiner (2008); D., Perrin (2019)]

Let $\mathcal{L}$ be a recurrent eventually neutral set with threshold m. For every $w \in \mathcal{L}^{\geq m}$, one has

$$\mathrm{Card}\left(\mathcal{R}(w)\right) = 1 + \sum_{|u|<m} \left(\mathrm{Card}\left(R(u)\right) - 1\right).$$

Cardinality of return words

Theorem [Vuillon (2001); Balková, Pelantová, Steiner (2008); D., Perrin (2019)]

Let $\mathcal{L}$ be a recurrent eventually neutral set with threshold m. For every $w \in \mathcal{L}^{\geq m}$, one has

$$\text{Card}\left(\mathcal{R}(w)\right) = 1 + \sum_{|u|<m} \left(\text{Card}\left(R(u)\right) - 1\right).$$

Corollary

An eventually neutral (dendric) set is recurrent if and only if it is uniformly recurrent

Proof. A recurrent set $\mathcal{L}$ is uniformly recurrent if and only if $\mathcal{R}(w)$ is finite for all $w \in \mathcal{L}$.

Open questions

- Is there a finite $\mathcal{S}$-adic representation for recurrent eventually dendric sets ?

  [When the set is *purely* dendric, there is one.]

- Subgroup generated by sets of return words in an eventually dendric set ?

  [For a dendric set, $\mathcal{R}(w)$ is a basis of the free group on $\mathcal{A}$.]

- Decidability of the (eventually) dendric condition.

  [Work in progress with Revekka Kyriakoglou and Julien Leroy]

Thank you