Tree sets

from Combinatorics on Words to Symbolic Dynamics

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Séminaire CANA

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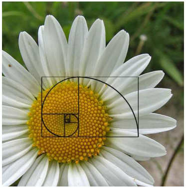
Fibonacci

$$x = abaababaabaababa\cdots$$

$$x = \lim_{n \to \infty} \varphi^n(a) \qquad \text{where} \qquad \varphi : \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$



Fibonacci

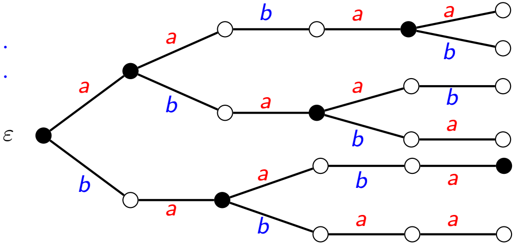
$x = abaababaabaababa\cdots$

The *Fibonacci set* (set of factors of x) is a Sturmian set.

Definition

A *Sturmian* set $S \subset \mathcal{A}^*$ is a factorial set such that $p_n = \mathrm{Card}\,(S \cap \mathcal{A}^n) = n + 1$.

n :	0	1	2	3	4	5	$\cdots$
p_n :	1	2	3	4	5	6	$\cdots$



2-coded Fibonacci

$$x = ab\ aa\ ba\ ba\ ab\ aa\ ba\ ba\ \cdots$$

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Arnoux-Rauzy sets

Definition

An *Arnoux-Rauzy* set is a factorial set closed by reversal with $p_n = (\mathrm{Card}\,(\mathcal{A}) - 1)n + 1$ having a unique right special factor for each length.

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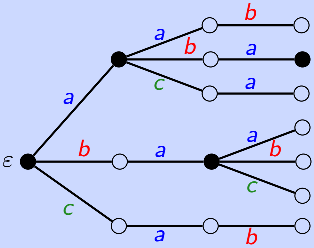
Example (Tribonacci)

Factors of the fixed point $\psi^\omega(a)$ of the morphism $\psi : a \mapsto ab, \quad b \mapsto ac, \quad c \mapsto a.$



n :	0	1	2	3	$\cdots$
p_n :	1	3	5	7	$\cdots$

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Is the set of factors of $f^{-1}(S)$ an Arnoux-Rauzy set ?

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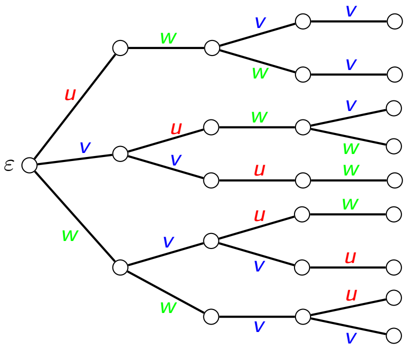

$$p_n = 2n + 1$$

n :	0	1	2	3	4	$\cdots$
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2-coded Fibonacci

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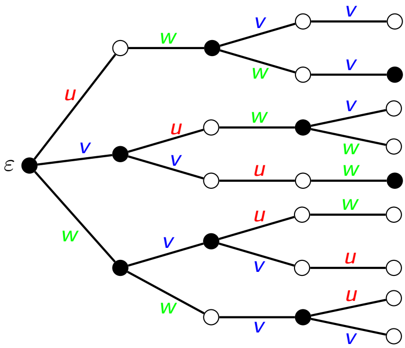
Is the set of factors of $f^{-1}(S)$ an Arnoux-Rauzy set ? No !



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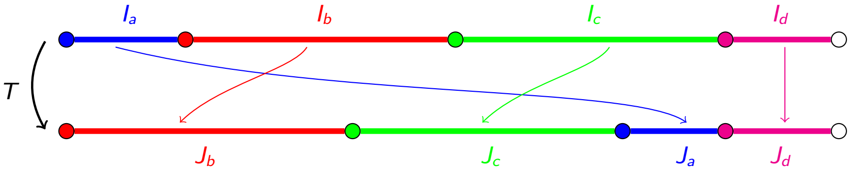
Interval exchanges

Let $(I_\alpha)_{\alpha\in\mathcal{A}}$ and $(J_\alpha)_{\alpha\in\mathcal{A}}$ be two partitions of $[0,1[$.
An *interval exchange transformation* (IET) is a map $T : [0,1[\to [0,1[$ defined by

$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha .$$



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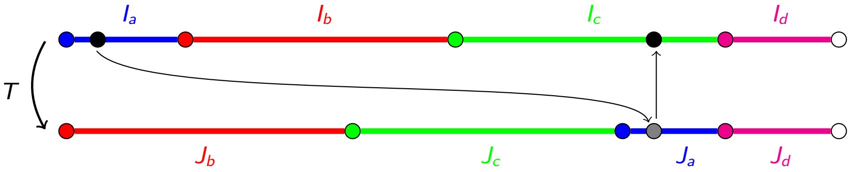

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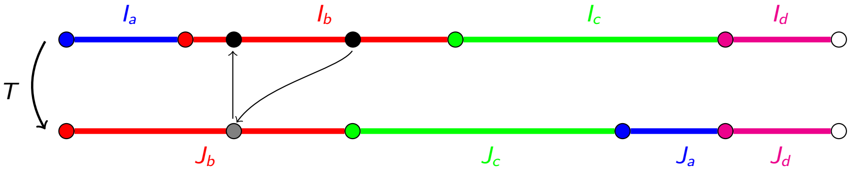

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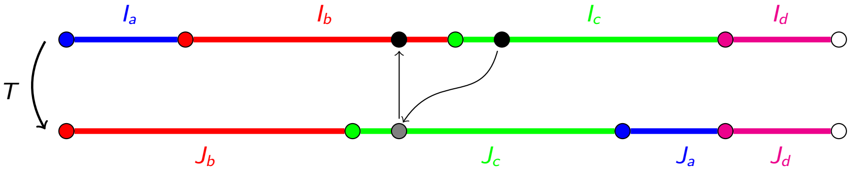

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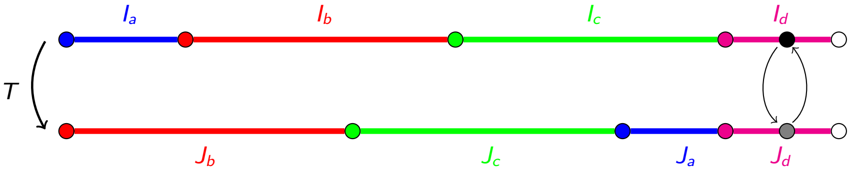

Interval exchanges

T is said to be *minimal* if for any point $z \in [0,1[$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[0,1[$.

T is said *regular* if the orbits of the non-zero separation points are infinite and disjoint.

Theorem [M. Keane (1975)]

A regular interval exchange transformation is minimal.

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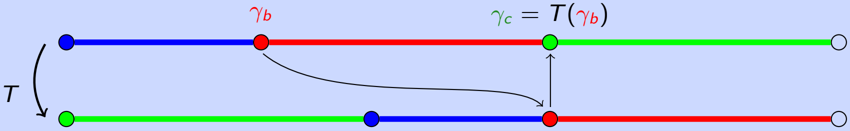
Example (the converse is not true)

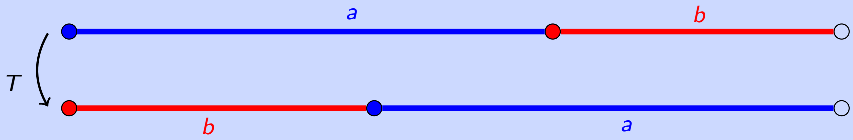


Interval exchanges

The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in \mathcal{A}^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha.$$

Example (Fibonacci, $z = (3-\sqrt{5})/2$)



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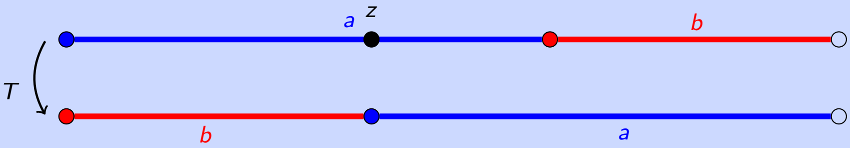

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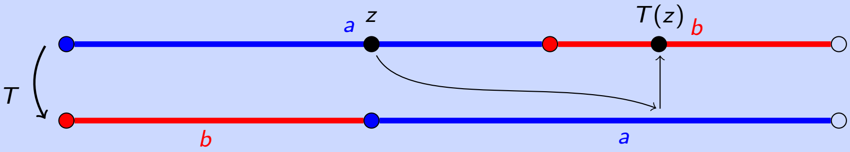

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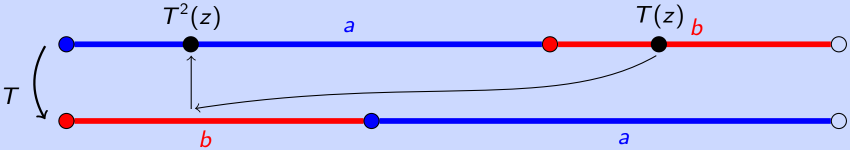

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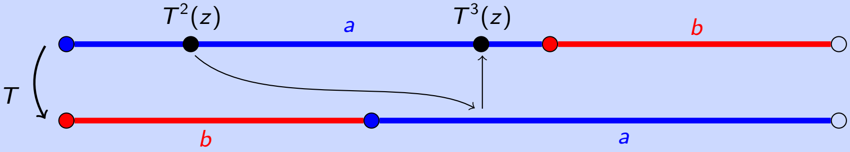

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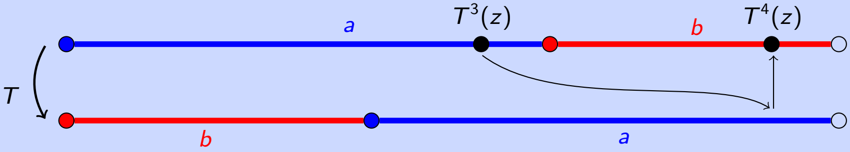

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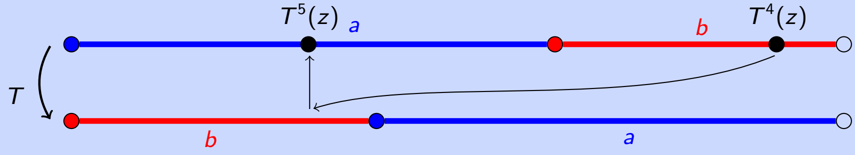

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The set $\mathcal{L}(T) = \bigcup_{z \in [0,1[} \mathrm{Fac}(\Sigma_T(z))$ is said a (*minimal*, *regular*) *interval exchange set*.

Remark. If T is minimal, $\mathrm{Fac}(\Sigma_T(z))$ does not depend on the point z.

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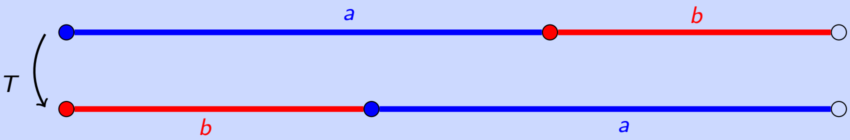

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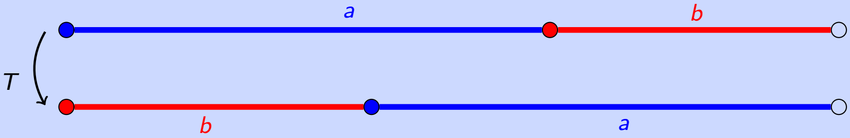
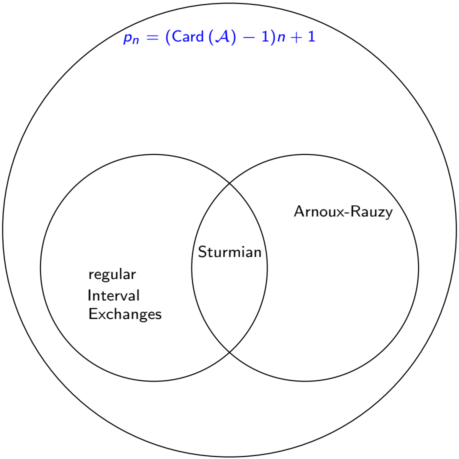


$$\mathcal{L}(T) = \left\{ \varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab\ \ldots \right\}$$

Proposition

Regular interval exchange sets have factor complexity $p_n = (\mathrm{Card}\,(\mathcal{A}) - 1)n + 1$.

Arnoux-Rauzy and Interval exchanges



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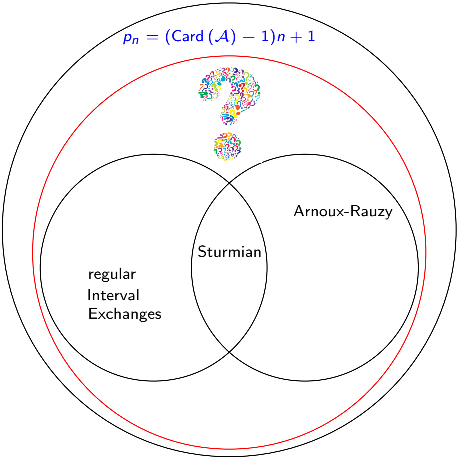

Extension graphs

The *extension graph* of a word $w \in S$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{a \in \mathcal{A} \,|\, aw \in S\}, \\ R(w) &= \{a \in \mathcal{A} \,|\, wa \in S\}, \\ B(w) &= \{(a, b) \in \mathcal{A} \times \mathcal{A} \,|\, awb \in S.\} \end{aligned}$$

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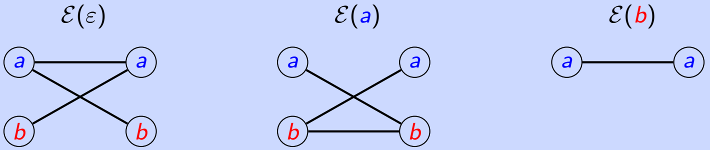

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The *multiplicity* of a word w is the quantity

$$m(w) = \text{Card}\,(B(w)) - \text{Card}\,(L(w)) - \text{Card}\,(R(w)) + 1.$$

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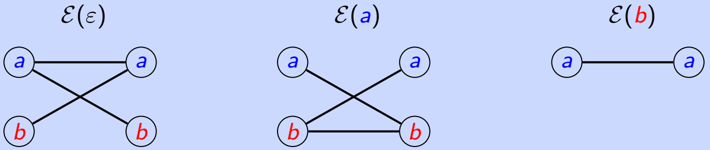
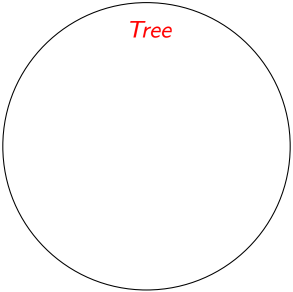


Tree and neutral sets

Definition

A factorial set S is called a *tree set* if the graph $\mathcal{E}(w)$ is a tree for any nonempty $w \in S$ and $\mathcal{E}(\varepsilon)$ a forest.



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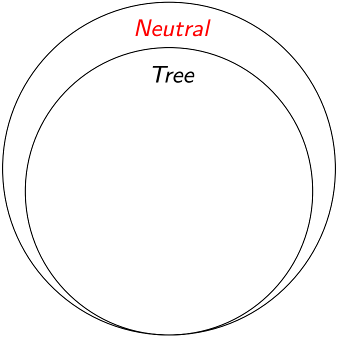

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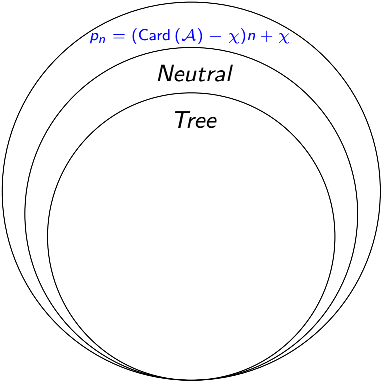
The *characteristic* of a neutral/tree set S is the quantity $\chi(S) = 1 - m(\varepsilon)$.



[*using* J. Cassaigne : "**Complexité et facteurs spéciaux**" (1997).]

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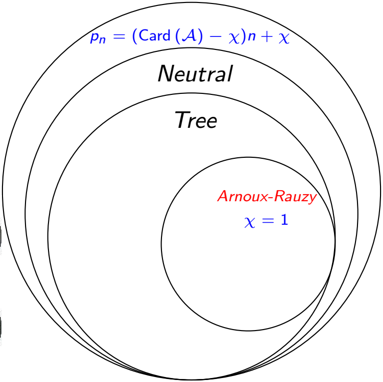

[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone : "**Acyclic, connected and tree sets**" (2014).]

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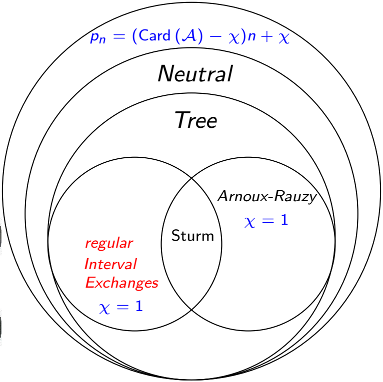

[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone : "**Bifix codes and interval exchanges**" (2015).]

Recurrence and uniformly recurrence

Definition

A factorial set S is *recurrent* if for every $u, v \in S$ there is a $w \in S$ such that uwv is in S.

Example (Fibonacci)

$$x = abaababaabaababaababaabaababa\cdots$$

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A factorial set S is *recurrent* if for every $u, v \in S$ there is a $w \in S$ such that uwv is in S.

It is *uniformly recurrent* (or *minimal*) if for every $u \in S$ there exists an $n \in \mathbb{N}$ such that u is a factor of every word of length n in S.

Example (Fibonacci)

$$x = \underbrace{abaa}_{4}\, ba\, \underbrace{baab}_{4}\, \underbrace{aaba}_{4}\, baababaaba\, \underbrace{abab}_{4}\, a \cdots$$

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Uniform recurrence $\implies$ recurrence.

Theorem [D., Perrin (2016)]

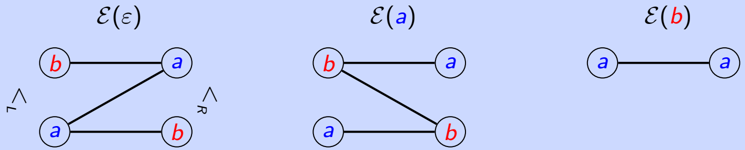
A recurrent neutral (tree) set is uniformly recurrent.

Planar tree sets

Let $<_L$ and $<_R$ be two orders on $\mathcal{A}$.
For a set S and a word $w \in S$, the graph $\mathcal{E}(w)$ is *compatible* with $<_L$ and $<_R$ if for any $(a, b), (c, d) \in B(w)$, one has

$$a <_L c \quad \Longrightarrow \quad b \leq_R d.$$

Example (Fibonacci, $b <_L a$ and $a <_R b$)

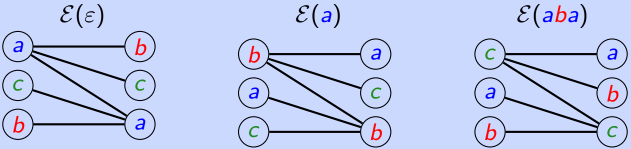


A biextendable set S is a *planar tree set* w.r.t. $<_L$ and $<_R$ on $\mathcal{A}$ if for any nonempty $w \in S$ (resp. ε) the graph $\mathcal{E}(w)$ is a tree (resp. forest) compatible with $<_L$ and $<_R$.

Planar tree sets

Example

The *Tribonacci set* is **not** a planar tree set.
Indeed, let us consider the extension graphs of the bispecial words ε, a and aba.



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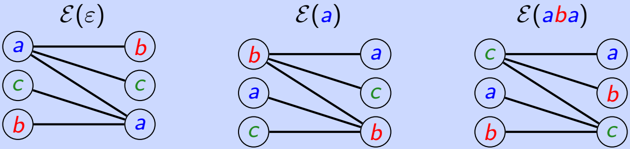
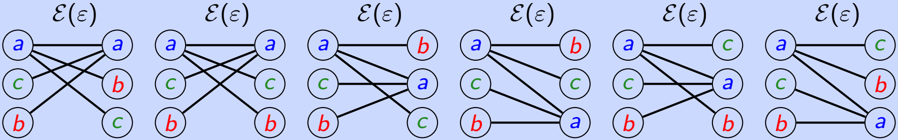


- $a <_L c <_L b$



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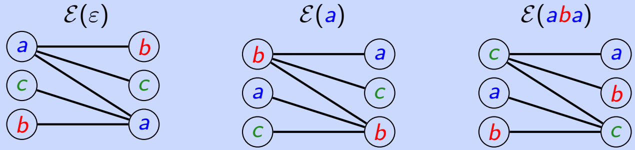
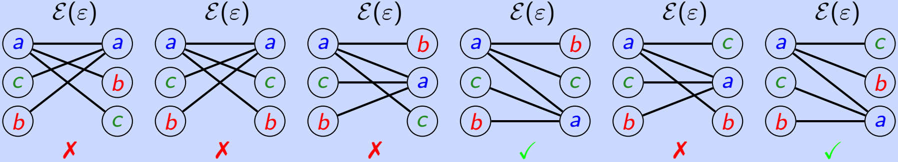


- $a <_L c <_L b \quad \Longrightarrow \quad b <_R c <_R a$ or $c <_R b <_R a$



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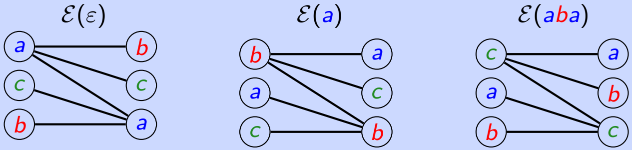

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Planar tree sets

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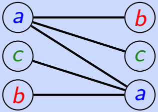
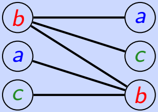
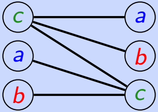


$\mathcal{E}(a)$



$\mathcal{E}(aba)$



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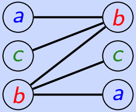

✓

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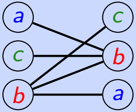

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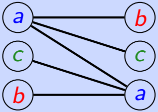

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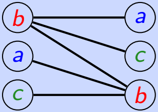

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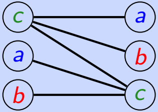

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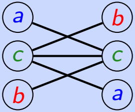

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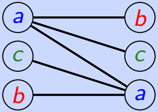

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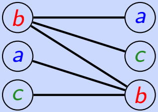

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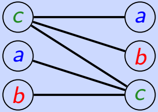

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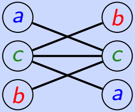
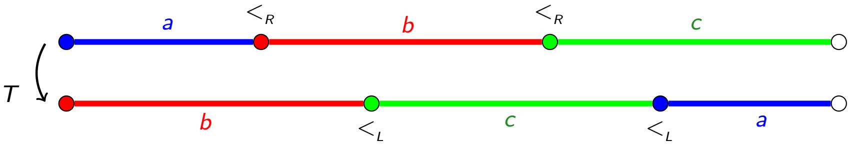
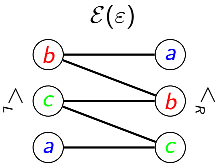


Planar tree sets

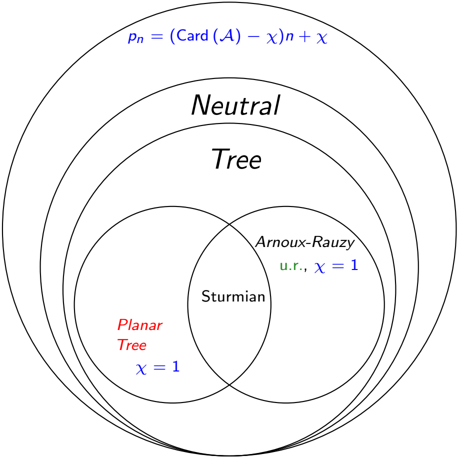
Theorem [S. Ferenczi, L. Zamboni (2008)]

A set S is a regular interval exchange set on $\mathcal{A}$ if and only if it is a recurrent planar tree set of characteristic 1.





Tree and neutral sets



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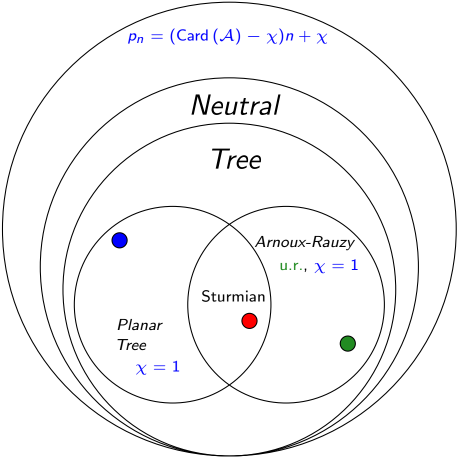
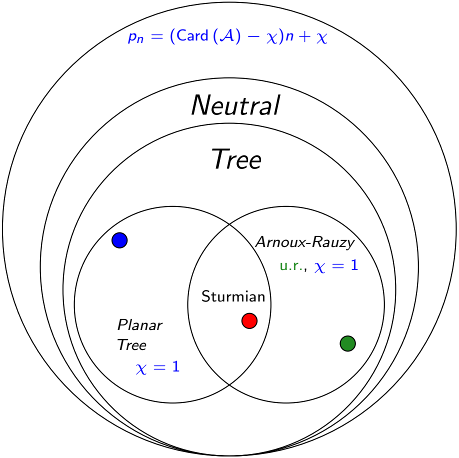


- Fibonacci
- Tribonacci
- regular IE

Tree and neutral sets



- ● Fibonacci
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- ? 2-coded Tribonacci
- ● regular IE
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Bifix codes

Definition

A *bifix code* is a set $B \subset \mathcal{A}^+$ of nonempty words that does not contain any proper prefix or suffix of its elements.

Example

- ✓ $\{aa, ab, ba\}$
- ✓ $\{aa, ab, bba, bbb\}$
- ✓ $\{ac, bcc, bcbca\}$
- ✗ { avril, mars, Marseille }
- ✗ { cap, calanque, que }
- ✗ { CANA, nada, Canada }

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A bifix code $B \subset S$ is *S-maximal* if it is not properly contained in a bifix code $C \subset S$.

Example (Fibonacci)

The set $B = \{aa, ab, ba\}$ is an S-maximal bifix code.
It is not an $\mathcal{A}^*$-maximal bifix code, since $B \subset B \cup \{bb\}$.

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A *coding morphism* for a bifix code $B \subset A^+$ is a morphism $f : \mathcal{B}^* \to \mathcal{A}^*$ which maps bijectively $\mathcal{B}$ onto B.

Example

The map $f : \{u, v, w\}^* \to \{a, b\}^*$ is a coding morphism for $B = \{aa, ab, ba\}$.

$$f : \begin{cases} u \mapsto aa \\ v \mapsto ab \\ w \mapsto ba \end{cases}$$

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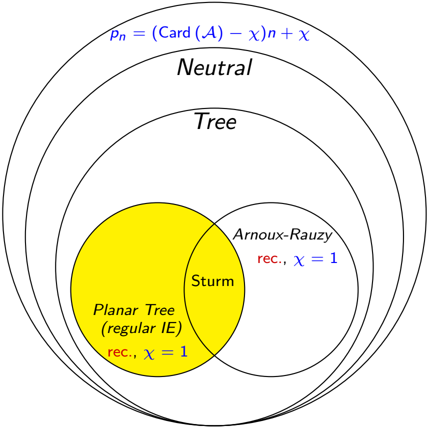
When S is factorial and B is an S-maximal bifix code, the set $f^{-1}(S)$ is called a *maximal bifix decoding* of S.

Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014)]

The family of recurrent planar tree sets of characteristic 1 (i.e. regular interval exchange sets) is closed under maximal bifix decoding.



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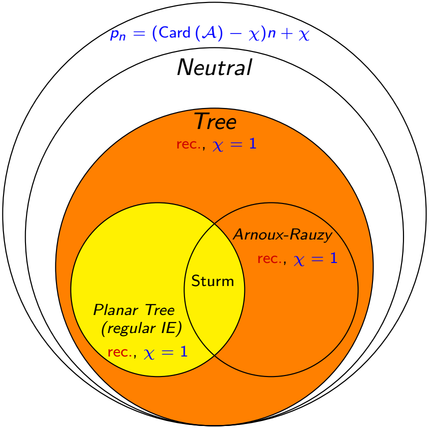
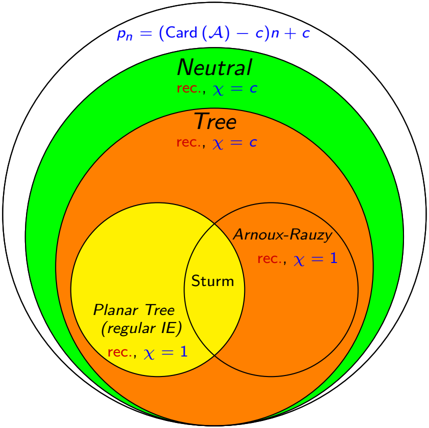


Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015) ; D., Perrin (2016)]

The family of recurrent neutral sets (resp. tree sets) of characteristic c is closed under maximal bifix decoding.



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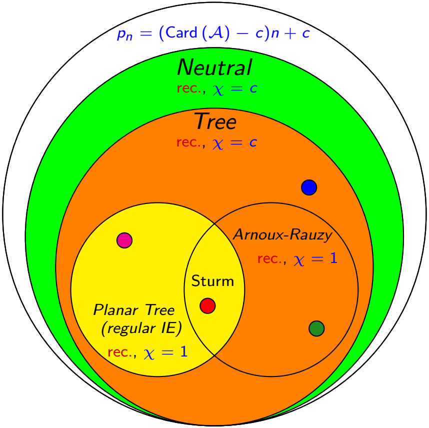

- Fibonacci
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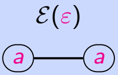
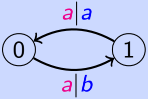
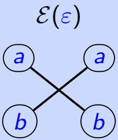
Tree sets of characteristic ≥ 1

Example (Multiplying transducer over a^*)







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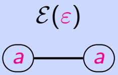
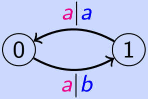
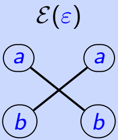
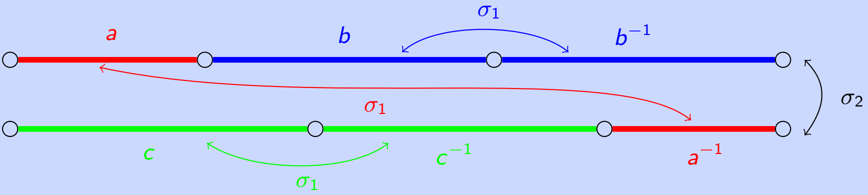






Example (Linear involution $T = \sigma_2 \circ \sigma_1$)



Tree subshifts

The *shift transformation* is the function

$$\begin{array}{rccc} \sigma : & \mathcal{A}^{\mathbb{Z}} & \to & \mathcal{A}^{\mathbb{Z}} \\ & (x_n)_{n\in\mathbb{Z}} & \mapsto & (x_{n+1})_{n\in\mathbb{Z}} \end{array}$$

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The pair (X, σ), with X a closed σ-invariant subset of $\mathcal{A}^{\mathbb{Z}}$ is called a *subshift*.

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(X, σ) is a *tree subshift* if its language $\mathcal{L}(X) = \bigcup_{\mathbf{x}\in X} \mathrm{Fac}(\mathbf{x})$ is a tree set.

Entropy of tree subshifts

The *entropy* of a shift (X, σ) having language $\mathcal{L}(X) \subset \mathcal{A}^*$ is defined as

$$h(X) = \lim_{n \to \infty} \frac{1}{n} \log(\mathcal{L}(X) \cap \mathcal{A}^n)$$

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Proposition

All tree subshifts have entropy zero.

Ergodicity of tree subshifts

A probability measure μ on (X, σ) is said to be *invariant* if $\mu\left(\sigma^{-1}(U)\right) = \mu(U)$ for every Borel subset U of X.

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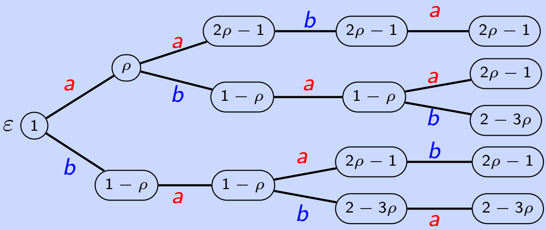
Theorem [P. Arnoux, G. Rauzy (1991)]

Subshifts associated to Arnoux-Rauzy sets are uniquely ergodic.

Example (Fibonacci, $\rho = (\sqrt{5} - 1)/2$)

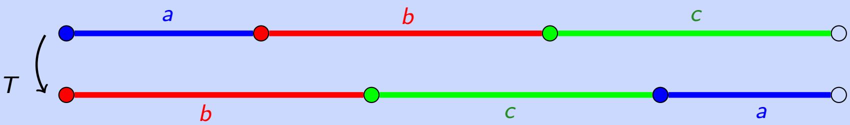


Ergodicity of tree subshifts

Given an interval exchange transformation T and a word $w = a_0 a_1 \cdots a_{m-1} \in \mathcal{A}^*$, let

$$I_w = I_{a_0} \cap T^{-1}(I_{a_1}) \cap \ldots \cap T^{-m+1}(I_{a_{m-1}})$$

Example



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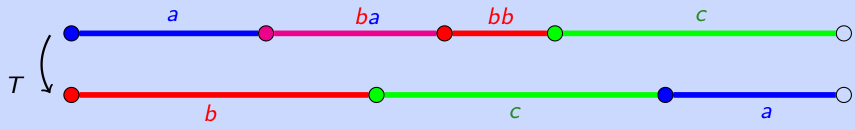

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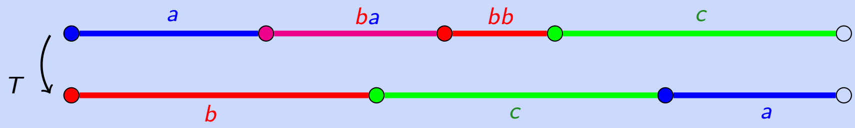

The map λ defined by $\lambda([w]) = |I_w|$ is an invariant probability measure.

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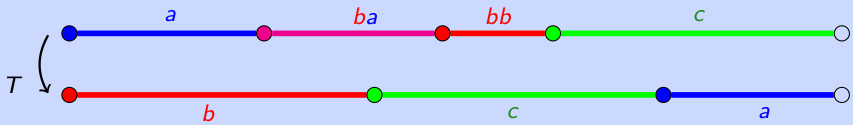

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QUESTION : Is it the only one?

Ergodicity of a tree subshift

Conjecture [M. Keane (1975)]

Every regular IE is uniquely ergodic.

Ergodicity of a tree subshift

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Theorem [H. Masur (1982), W. Veech (1982)]

Almost all regular IE are uniquely ergodic.

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Corollary

Tree subshift are **not** in general uniquely ergodic (even when minimal).

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A minimal symbolic system such that $\limsup_{n\to\infty} \left(\frac{p_n}{n}\right) < 3$ is uniquely ergodic.

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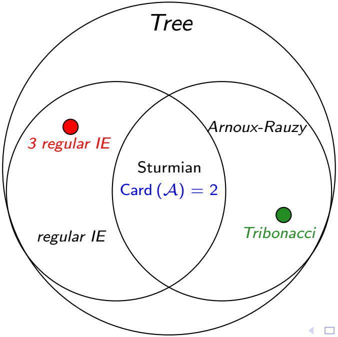

Minimal tree subshifts on a 3-letter alphabet

Two subshifts $(X, \sigma), (Y, \sigma)$ are *orbit equivalent* if there exists a homeomorphism $\eta : X \to Y$ such that for all $\mathbf{x} \in X$ one has

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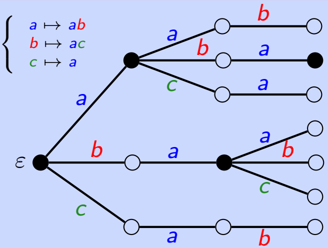
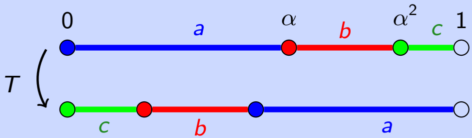
Theorem [V. Berthé, P. Cecchi, F.D., F. Durand, J. Leroy, D. Perrin, S. Petite (2018+)]

All minimal tree subshifts on a 3 letter alphabet having the same letter frequency are orbit equivalent.

Example $(\alpha + \alpha^2 + \alpha^3 = 1)$

$$\psi : \begin{cases} a \mapsto ab \\ b \mapsto ac \\ c \mapsto a \end{cases}$$






Merci