On balanced sequences and their asymptotic critical exponent

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Outline of the talk

- **Balanced Sequences and Where to Find Them**
  (*What they are, how to construct them and some interesting properties*)

- **Repetita iuvant**
  (*Critical exponent and asympotic critical exponent*)

- **Homework**
  (*Perspectives and open problems*)

A few words about words

- $\mathcal{A}$ is an *alphabet*, and its elements $a \in \mathcal{A}$ are *letters*.
- $\mathcal{A}^* := \{w = a_1 a_2 \cdots a_n \mid n \geq 0,\ a_i \in \mathcal{A}\}$ is the set of (finite) *words*.
- $\mathcal{A}^{\mathbb{N}} := \{\mathbf{w} = a_1 a_2 a_3 \cdots \mid a_i \in \mathcal{A}\}$ is the set of (infinite) *sequences*.
- $\mathbf{w} = uv^{\omega} = uvvv \cdots$ is *eventually periodic* (just *periodic* if $u = \varepsilon$).
  If $\mathbf{w}$ is not eventually periodic then it is *aperiodic*.
- $\mathcal{L}(w) := \{u \mid w = pus, \textit{with } p, s \in \mathcal{A}^*\}$ is the *language* of w.
  Similar definition for $\mathcal{L}(\mathbf{w})$.
- $|w|$ is the length of w; $|w|_a$ the number of occurrences of a in w.

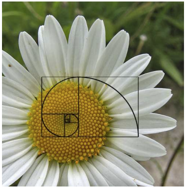
Fibonacci

$$\mathsf{x} = \mathtt{abaababaabaababa}\cdots$$

$$\mathsf{x} = \lim_{n\to\infty} \varphi^n(\mathtt{a}) \qquad \text{where} \qquad \varphi : \begin{cases} \mathtt{a} \mapsto \mathtt{ab} \\ \mathtt{b} \mapsto \mathtt{a} \end{cases}$$

The *Fibonacci language* is the set

$$\mathcal{L}(\mathsf{x}) = \{\varepsilon, \mathtt{a}, \mathtt{b}, \mathtt{aa}, \mathtt{ab}, \mathtt{ba}, \mathtt{aab}, \mathtt{aba}, \mathtt{baa}, \mathtt{bab}, \mathtt{aaba}, \ldots\}$$



Recurrence and uniform recurrence

Definition

A sequence $\mathbf{x}$ is *recurrent* if for every $u, v \in \mathcal{L}(\mathbf{x})$ there exists $w \in \mathcal{L}(\mathbf{x})$ such that $uwv \in \mathcal{L}(\mathbf{x})$.

Example (Fibonacci)

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Recurrence and uniform recurrence

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It is *uniformly recurrent* if for every $u \in \mathcal{L}(\mathbf{x})$ there exists an $n \in \mathbb{N}$ such that u is a factor of every word of length n of $\mathbf{x}$.

Example (Fibonacci)

$$\mathbf{x} = \underbrace{\mathtt{abaa}}_{4}\,\mathtt{ba}\,\underbrace{\mathtt{baab}}_{4}\,\underbrace{\mathtt{aaba}}_{4}\,\mathtt{baababaaba}\,\underbrace{\mathtt{abab}}_{4}\,\mathtt{a}\cdots$$

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Proposition

Uniform recurrence $\Longrightarrow$ Recurrence.

Balanced sequences

A word $w \in \mathcal{A}^*$ (resp. sequence $\mathbf{w} \in \mathcal{A}^{\mathbb{N}}$) is *balanced* if for every $a \in \mathcal{A}$ and every $u, v \in \mathcal{L}(w)$ (resp. $u, v \in \mathcal{L}(\mathbf{w})$) with $|u| = |v|$ we have

$$|u|_a - |v|_a \leq 1.$$

Example

- 0100101 is balanced.
- 0100011 is not balanced, indeed $|000|_1 = 0$ and $|011|_1 = 2$.

Balance sequence (1 portion)

INGREDIENTS:

- 1 Sturmian sequence $\mathbf{x}$ over $\{\mathtt{a}, \mathtt{b}\}$;
- 2 constant gap sequences $\mathbf{y}$ and $\mathbf{y}'$ over disjoint $\mathcal{A}$ and $\mathcal{B}$.

DIRECTIONS:

- Replace every $\mathtt{a}$ by $\mathbf{y}$ and every $\mathtt{b}$ by $\mathbf{y}'$.

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Directions:

- Replace every a by $\mathbf{y}$ and every b by $\mathbf{y}'$.

Sturmian sequences

First ingredient

Definition

A sequence $\mathbf{x}$ is *Sturmian* if $\mathcal{C}_{\mathbf{x}}(n) = \mathrm{Card}\,(\mathcal{L}(\mathbf{x}) \cap \mathcal{A}^n) = n + 1$.

Example (Fibonacci)

$$\mathcal{L}(\mathbf{x}) = \{\underbrace{\varepsilon}_{1}, \underbrace{\mathtt{a}, \mathtt{b}}_{2}, \underbrace{\mathtt{aa}, \mathtt{ab}, \mathtt{ba}}_{3}, \underbrace{\mathtt{aab}, \mathtt{aba}, \mathtt{baa}, \mathtt{bab}}_{4}, \underbrace{\mathtt{aaba}, \mathtt{abaa}, \mathtt{abab}, \mathtt{baab}, \mathtt{baba}}_{5}, \ldots\}$$

$$\mathbf{x} = \lim_{n\to\infty} \Delta_0 \Delta_1 \cdots \Delta_n \left(\mathbf{u}^{(n)}\right) \qquad \Delta = D^{a_0} G^{a_1} D^{a_2} G^{a_3} \cdots$$ *directive sequence*

$$\theta = [a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

Constant gap sequences

Second ingredient

Definition

A sequence $\mathbf{y} \in \mathcal{A}^{\mathbb{N}}$ is a *constant gap sequence* if for each $a \in \mathcal{A}$ appearing in $\mathbf{y}$ there is a positive integer d such that the distance between two succesive occurences of a in $\mathbf{y}$ is d.

Examples

- $(0102)^{\omega}$ and $(34)^{\omega}$ are constant gap sequences.
- $(011)^{\omega}$ is periodic but NOT a constant gap sequence.

We denote by $\mathrm{Per}(\mathbf{y})$ the minimal period of $\mathbf{y}$.

Balanced sequences

Cooking time

Theorem [Graham (1973), Hubert (2000)]

A recurrent aperiodic sequence $\mathbf{v}$ is balanced if and only if $\mathbf{v} = \mathrm{colour}(\mathbf{x}, \mathbf{y}, \mathbf{y}')$ with $\mathbf{x}$ Sturmian and $\mathbf{y}, \mathbf{y}'$ constant gap sequences over disjoint aphabets.

Example (colouring of Fibonacci)

$$\mathbf{x} = abaababaabaababa\cdots$$

$$\mathbf{y} = (0102)^\omega \qquad\qquad \mathbf{y}' = (34)^\omega$$

$$\mathbf{v} = \mathrm{colour}(\mathbf{x}, \mathbf{y}, \mathbf{y}') = 0310423014023041\cdots$$

Remark. Sturmian sequences are balanced ($\mathbf{y} = (0)^\omega$, $\mathbf{y}' = (1)^\omega$).

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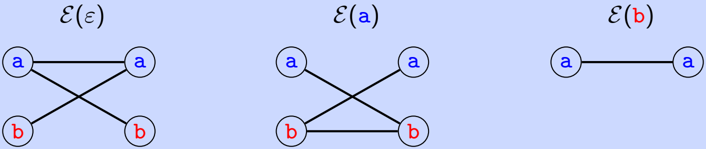
What's special about balanced sequences?

Extension graphs

The *extension graph* of a word $w \in \mathcal{L}$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{u \in \mathcal{A} \,|\, uw \in \mathcal{L}\} \\ R(w) &= \{v \in \mathcal{A} \,|\, wv \in \mathcal{L}\} \\ B(w) &= \{(u,v) \in \mathcal{A} \times \mathcal{A} \,|\, uwv \in \mathcal{L}\} \end{aligned}$$

Example (Fibonacci, $\mathcal{L}(\mathrm{x}) = \{\varepsilon, \mathrm{a}, \mathrm{b}, \mathrm{aa}, \mathrm{ab}, \mathrm{ba}, \mathrm{aab}, \mathrm{aba}, \mathrm{baa}, \mathrm{bab}, \ldots\}$)



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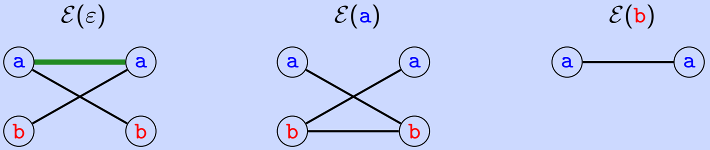

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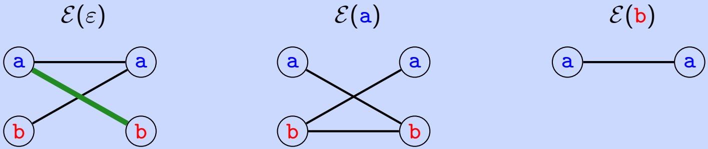

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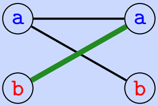
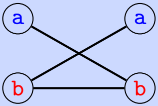
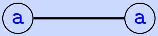
$\mathcal{E}(\varepsilon)$



$\mathcal{E}(\mathtt{a})$



$\mathcal{E}(\mathtt{b})$



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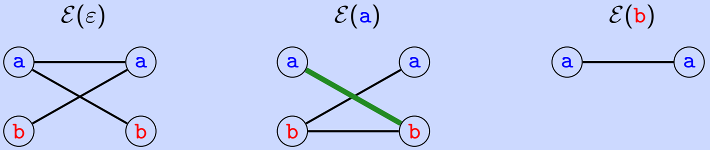

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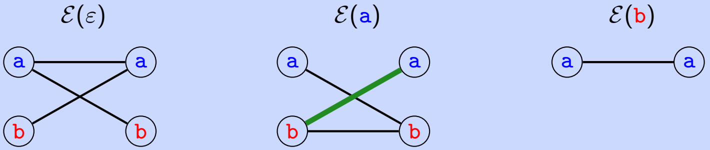

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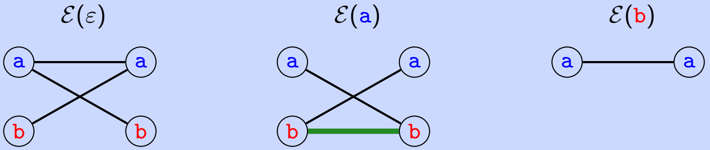

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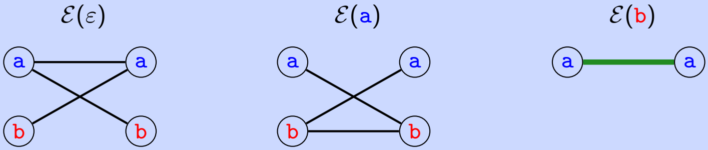

Dendric languages

Definition

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Example (Fibonacci)

$\mathcal{R}_{\mathbf{x}}(\mathrm{b}) = \{\mathrm{ab}, \mathrm{aab}\}$ $\qquad$ $\mathbf{x} = \mathrm{abaababaabaababaababaabaababaabaab}\cdots$

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- recurrence $\iff$ uniformly recurrence.

Repetitions

- A *period* of a word w is an integer p s.t. $a_i = a_{i+p}$ for all i.

Example

$w =$ aabaabaa $\qquad p = 3$

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- w has *fractional root* u and *exponent* $e = \frac{|w|}{|u|}$ if $w = u^e$.

Example

$w = \mathrm{aabaabaa}$ $\qquad p = 3$ $\qquad w = (\mathrm{aab})^{8/3}$ $\qquad e = \frac{8}{3}$

$$w = \mathrm{aab.aab.aa}$$

Avoiding repetitions

Theorem

1. Squares are unavoidable over a binary alphabet.

Proof.

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Avoiding repetitions

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2. Squares are avoidable over a ternary alphabet.

Proof.

1. aba**?**
2. $\psi : \begin{cases} \mathtt{a} \mapsto \mathtt{abc} \\ \mathtt{b} \mapsto \mathtt{ac} \\ \mathtt{c} \mapsto \mathtt{b} \end{cases}$ $\psi^{\omega}(\mathtt{a}) = \mathtt{abcacbabcbacac}\cdots$ is square-free. $(e < 2)$

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Theorem

1. Squares are unavoidable over a binary alphabet.
2. Squares are avoidable over a ternary alphabet.
3. Cubes are avoidable over a binary alphabet.

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3. $\tau : \begin{cases} \mathtt{a} \mapsto \mathtt{ab} \\ \mathtt{b} \mapsto \mathtt{ba} \end{cases}$ $\tau^\omega(\mathtt{a}) = \mathtt{abbabaabbaabab}\cdots$ is cube-free. $(e < 3)$

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2. $\psi : \begin{cases} a \mapsto abc \\ b \mapsto ac \\ c \mapsto b \end{cases}$ $\psi^\omega(a) = abcacbabcbacac\cdots$ is square-free. $(e < 2)$
3. $\tau : \begin{cases} a \mapsto ab \\ b \mapsto ba \end{cases}$ $\tau^\omega(a) = abba\,\underbrace{baab}\,\underbrace{baab}\,ab\cdots$ is overlap-free. $(e < 2 + \epsilon)$

(Asymptotic) critical exponent

Definition

The *critical exponent* of a sequence $\mathbf{u}$ is

$$\mathsf{E}(\mathbf{u}) = \sup \left\{ e \in \mathbb{Q} : w^e \in \mathcal{L}(\mathbf{u}), \text{with } w \neq \varepsilon \right\}$$

The *asymptotic critical exponent* of a sequence $\mathbf{u}$ is

$$\mathsf{E}^*(\mathbf{u}) = \lim_{n \to \infty} \left(\sup \left\{ e \in \mathbb{Q} : w^e \in \mathcal{L}(\mathbf{u}), \text{with } |w| > n \right\} \right)$$

Remark. $\mathsf{E}(\mathbf{u}) \geq \mathsf{E}^*(\mathbf{u})$.

Dejean's conjecture

Theorem (*Dejean's conjecture* (1972)) [Currie, Rampersad (2009), Rao (2009)]

The least critical exponent of a sequence over an alphabet of size d is

$$\begin{cases} 7/4 & d = 3 \\ 7/5 & d = 4 \\ \frac{d}{d-1} & d = 2 \text{ or } d \geq 5 \end{cases}$$

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What about balanced sequences?

Critical exponent of balanced sequences

Theorem [Mignosi, Pirillo (2001), Peltomäki (2019), Rampersad, Shallit, Vandomme (2019)]

The least critical exponent of a balanced sequence over an alphabet of size d is

$$\begin{cases} 1+\frac{1+\sqrt{5}}{2} & d=2 \\ 2+\frac{\sqrt{2}}{2} & d=3 \\ 1+\frac{1+\sqrt{5}}{4} & d=4 \\ \frac{d-2}{d-3} & 5\leq d\leq 8 \end{cases}$$

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$$\begin{cases} 1+\frac{1+\sqrt{5}}{2} & d=2 \\ 2+\frac{\sqrt{2}}{2} & d=3 \\ 1+\frac{1+\sqrt{5}}{4} & d=4 \\ \frac{d-2}{d-3} & 5 \leq d \leq 8 \end{cases}$$

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Theorem [D., Dvořáková, Pelantová (2021)]

- True for $d = 9, 10$.

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Theorem [D., Dvořáková, Pelantová (2021, 2021+)]

- True for $d = 9, 10$.
- **False**, new bound $\frac{d-1}{d-2}$ for $d = 11, 12$.

(Asymptotic) critical exponent of balanced sequences

Proposition [D., Dvořáková, Pelantová (2021)]

Let $\mathbf{u}$ be a uniformly recurrent aperiodic sequence. Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of all bispecial factors ordered by their length, and v_n a shortest return word to w_n in $\mathbf{u}$. Then

$$E(\mathbf{u}) = 1 + \sup_{n\in\mathbb{N}} \left\{ \frac{|w_n|}{|v_n|} \right\} \qquad \text{and} \qquad E^*(\mathbf{u}) = 1 + \limsup_{n\to\infty} \frac{|w_n|}{|v_n|}.$$

Main result

Theorem [D., Dvořáková, Pelantová (2021)]

Let $\mathsf{v} = \mathrm{colour}(\mathsf{u}, \mathsf{y}, \mathsf{y}')$, with u Sturmian and y, y' constant gap sequences. Then

$$\mathsf{E}(\mathsf{v}) \geq \mathsf{E}^*(\mathsf{v}) \geq 1 + \frac{1}{\mathrm{Per}(\mathsf{y})\mathrm{Per}(\mathsf{y}')} .$$

Moreover, $\mathsf{E}^*(\mathsf{v})$ depends only on $\mathrm{Per}(\mathsf{y})$ and $\mathrm{Per}(\mathsf{y}')$ (not on the structure of y and y').

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Let $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$, with $\mathbf{u}$ Sturmian and $\mathbf{y}, \mathbf{y}'$ constant gap sequences. Then

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Moreover, $\mathsf{E}^*(\mathbf{v})$ depends only on $\mathrm{Per}(\mathbf{y})$ and $\mathrm{Per}(\mathbf{y}')$ (not on the structure of $\mathbf{y}$ and $\mathbf{y}'$).

Program [with Opočenská (2021)]

Input: θ (quadratic irrational), $\mathrm{Per}(\mathbf{y})$ and $\mathrm{Per}(\mathbf{y}')$.
Output: $\mathsf{E}^*(\mathbf{u})$.

Main result

d	θ	$\mathbf{y}$	$\mathbf{y}'$	$E(\mathbf{v})$	$E^*(\mathbf{v})$
2	$[0;\overline{1}]$	0^ω	1^ω	$1+\frac{1+\sqrt{5}}{\sqrt{2}}$	$1+\frac{1+\sqrt{5}}{\sqrt{2}}$
3	$[0;1,\overline{2}]$	0^ω	$(12)^\omega$	$2+\frac{1}{\sqrt{2}}$	$2+\frac{1}{\sqrt{2}}$
4	$[0;\overline{1}]$	$(01)^\omega$	$(23)^\omega$	$1+\frac{1+\sqrt{5}}{4}$	$1+\frac{1+\sqrt{5}}{4}$
5	$[0;1,\overline{2}]$	$(01)^\omega$	$(2324)^\omega$	$\frac{3}{2}$	$\frac{3}{2}$
6	$[0;2,1,1,\overline{1,1,1,2}]$	0^ω	$(123415321435)^\omega$	$\frac{4}{3}$	$\frac{4}{3}$
7	$[0;1,3,\overline{1,2,1}]$	$(01)^\omega$	$(234526432546)^\omega$	$\frac{5}{4}$	$\frac{5}{4}$
8	$[0;3,1,\overline{2}]$	$(01)^\omega$	$(2345267325462375264325 76)^\omega$	$\frac{6}{5}=1.2$	$\frac{12+3\sqrt{2}}{14}\doteq 1.16$
9	$[0;2,3,\overline{2}]$	$(01)^\omega$	$(234567284365274863254768)^\omega$	$\frac{7}{6}\doteq 1.167$	$1+\frac{2\sqrt{2}-1}{14}\doteq 1.131$
10	$[0;4,2,\overline{3}]$	$(01)^\omega$	$(2345672849632547682943652748 69)^\omega$	$\frac{8}{7}\doteq 1.14$	$1+\frac{\sqrt{13}}{26}\doteq 1.139$

Main result

d	θ	$\mathbf{y}$	$\mathbf{y}'$	$\mathsf{E}(\mathbf{v})$	$\mathsf{E}^*(\mathbf{v})$
2	$[0;\overline{1}]$	0^ω	1^ω	$1+\frac{1+\sqrt{5}}{\sqrt{2}}$	$1+\frac{1+\sqrt{5}}{\sqrt{2}}$
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6	$[0;2,1,1,\overline{1,1,1,2}]$	0^ω	$(123415321435)^\omega$	$\frac{4}{3}$	$\frac{4}{3}$
7	$[0;1,3,\overline{1,2,1}]$	$(01)^\omega$	$(234526432546)^\omega$	$\frac{5}{4}$	$\frac{5}{4}$
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- $\mathsf{E}^*(\mathbf{x}_d)=\mathsf{E}(\mathbf{x}_d)$ for $d=3,4,5,6,7$.

Main result

d	θ	$\mathbf{y}$	$\mathbf{y}'$	$E(\mathbf{v})$	$E^*(\mathbf{v})$
2	$[0;\overline{1}]$	0^ω	1^ω	$1+\frac{1+\sqrt{5}}{\sqrt{2}}$	$1+\frac{1+\sqrt{5}}{\sqrt{2}}$
3	$[0;1,\overline{2}]$	0^ω	$(12)^\omega$	$2+\frac{1}{\sqrt{2}}$	$2+\frac{1}{\sqrt{2}}$
4	$[0;\overline{1}]$	$(01)^\omega$	$(23)^\omega$	$1+\frac{1+\sqrt{5}}{4}$	$1+\frac{1+\sqrt{5}}{4}$
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7	$[0;1,3,\overline{1,2,1}]$	$(01)^\omega$	$(234526432546)^\omega$	$\frac{5}{4}$	$\frac{5}{4}$
8	$[0;3,1,\overline{2}]$	$(01)^\omega$	$(234526732546237526432576)^\omega$	$\frac{6}{5}=1.2$	$\frac{12+3\sqrt{2}}{14}\doteq 1.16$
9	$[0;2,3,\overline{2}]$	$(01)^\omega$	$(234567284365274863254768)^\omega$	$\frac{7}{6}\doteq 1.167$	$1+\frac{2\sqrt{2}-1}{14}\doteq 1.131$
10	$[0;4,2,\overline{3}]$	$(01)^\omega$	$(2345672849632547682943652748 69)^\omega$	$\frac{8}{7}\doteq 1.14$	$1+\frac{\sqrt{13}}{26}\doteq 1.139$

- $E^*(\mathbf{x}_d) = E(\mathbf{x}_d)$ for $d = 3, 4, 5, 6, 7$.
- There exists $\mathbf{x} \in \{0, \ldots 7\}^*$ s.t. $E^*(\mathbf{x}) = E^*(\mathbf{x}_9) < E^*(\mathbf{x}_8)$.

Main result

d	θ	$\mathbf{y}$	$\mathbf{y}'$	$\mathsf{E}(\mathbf{v})$	$\mathsf{E}^*(\mathbf{v})$
2	$[0;\overline{1}]$	0^ω	1^ω	$1+\frac{1+\sqrt{5}}{\sqrt{2}}$	$1+\frac{1+\sqrt{5}}{\sqrt{2}}$
3	$[0;1,\overline{2}]$	0^ω	$(12)^\omega$	$2+\frac{1}{\sqrt{2}}$	$2+\frac{1}{\sqrt{2}}$
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6	$[0;2,1,1,\overline{1,1,1,2}]$	0^ω	$(123415321435)^\omega$	$\frac{4}{3}$	$\frac{4}{3}$
7	$[0;1,3,\overline{1,2,1}]$	$(01)^\omega$	$(234526432546)^\omega$	$\frac{5}{4}$	$\frac{5}{4}$
8	$[0;3,1,\overline{2}]$	$(01)^\omega$	$(2345267325462375264325\,76)^\omega$	$\frac{6}{5}=1.2$	$\frac{12+3\sqrt{2}}{14}\doteq 1.16$
9	$[0;2,3,\overline{2}]$	$(01)^\omega$	$(234567284365274863254768)^\omega$	$\frac{7}{6}\doteq 1.167$	$1+\frac{2\sqrt{2}-1}{14}\doteq 1.131$
10	$[0;4,2,\overline{3}]$	$(01)^\omega$	$(2345672849632547682943652748 69)^\omega$	$\frac{8}{7}\doteq 1.14$	$1+\frac{\sqrt{13}}{26}\doteq 1.139$
11	$[0;5,1,\overline{1,1,1,2}]$	$(01)^\omega$	$(234567892\mathrm{A}436587294\mathrm{A}638527496\mathrm{A}832547698\mathrm{A})^\omega$	$\frac{10}{9}\doteq 1.11$	$\frac{415+5\sqrt{105}}{424}\doteq 1.0996$
12	$[0;1,3,\overline{2}]$	$(012345)^\omega$	$(6789\mathrm{AB})^\omega$	$\frac{11}{10}=1.1$	$\frac{8-\sqrt{2}}{6}\doteq 1.0976$

- $\mathsf{E}^*(\mathbf{x}_d)=\mathsf{E}(\mathbf{x}_d)$ for $d=3,4,5,6,7$.
- There exists $\mathbf{x}\in\{0,\dots 7\}^*$ s.t. $\mathsf{E}^*(\mathbf{x})=\mathsf{E}^*(\mathbf{x}_9)<\mathsf{E}^*(\mathbf{x}_8)$.
- $\frac{d-1}{d-2}<\frac{d-2}{d-3}$

Open problems

- Find new candidates $\mathbf{x}_d$ with minimal (asymptotic) critical exponent, for $d \geq 13$.

- $\frac{d-2}{d-3} > \frac{d-1}{d-2} > \; ? \; \geq \frac{d}{d-1}$

- **Conjecture**: For $d = 2k$, with $k \geq 1$, the least $E^*(\mathbf{v})$ is reached for $\mathbf{v} = \mathrm{colour}(\mathbf{x}, \mathbf{y}, \mathbf{y}')$ with $\mathbf{x}$ Fibonacci, $\mathrm{Per}(\mathbf{y}) = \mathrm{Per}(\mathbf{y}') = 2^{k-1}$. Then

$$E^*(\mathbf{v}) = 1 + \frac{1}{2^{k-1}\tau^s},$$

where s is a minimal integer satisfying $2^{k-1} < \tau^{s+3}$.

Grazie per l'attenzione