Specular sets

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Joint work with
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J. Leroy, D. Perrin, C. Reutenauer, G. Rindone

Introduction

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Introduction of new concepts : *parity* of words (*odd* and *even* words).

Some results about return words and palindromes.

Outline

Given an involution $\theta : A \to A$ (possibly with some fixed point), let us define

$$G_\theta = \langle a \in A \mid a \cdot \theta(a) = 1 \text{ for every } a \in A \rangle.$$

$G_\theta = \mathbb{Z}^i * (\mathbb{Z}/2\mathbb{Z})^j$ is a *specular group* of type (i, j), and $\mathrm{Card}\,(A) = 2i + j$ is its *symmetric rank*.

Example

Let $\theta : b \leftrightarrow d$ fixing a, c.

$$G_\theta = \langle a, b, c, d \mid a^2 = c^2 = bd = db = 1 \rangle.$$

$G_\theta = \mathbb{Z} * (\mathbb{Z}/2\mathbb{Z})^2$ is a specular group of type $(1, 2)$ and symmetric rank 4.

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Theorem [*using* Kurosh Subgroup Theorem]

Any subgroup of a specular group is specular.

A word is *θ-reduced* if it has no factor of the form $a\theta(a)$ for $a \in A$.

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The θ-reduction of the word $d\not{a}\not{a}ac\not{b}\not{d}$ is dac

A subset of a group G is called *symmetric* if it is closed under taking inverses (under θ).

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The set $X = \{a, adc, b, cba, d\}$ is symmetric, for $\theta : b \leftrightarrow d$ fixing a, c.

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The set $X = \{a, adc, b, cba, d\}$ is symmetric, for $\theta : b \leftrightarrow d$ fixing a, c.

A set X in a specular group G is called a *monoidal basis* of G if :

- it is symmetric ;
- the monoid that it generates is G ;
- any product $x_1 x_2 \cdots x_m$ such that $x_k x_{k+1} \neq 1$ for every k is distinct of 1.

Example

The alphabet A is a monoidal basis of G_θ.

The *symmetric rank* of a specular group is the cardinality of any monoidal basis.

The *extension graph* of a word $w \in S$ is the undirected bipartite graph $G(w)$ with vertices the disjoint union of

$$L(w) = \{a \in A \mid aw \in S\} \quad \text{and} \quad R(w) = \{a \in A \mid wa \in S\},$$

and edges the pairs $E(w) = \{(a,b) \in A \times A \mid awb \in S\}$.

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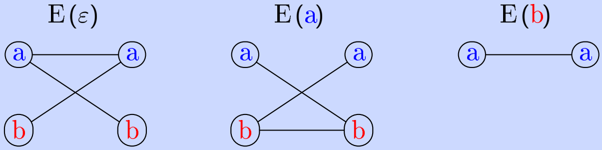
Example (Fibonacci)

$S = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, bab, \ldots\}$.



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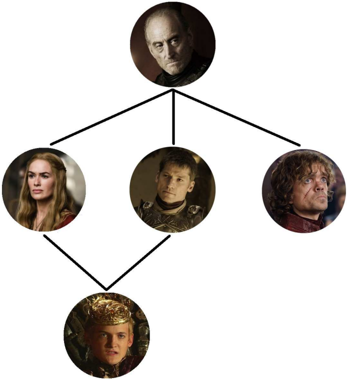

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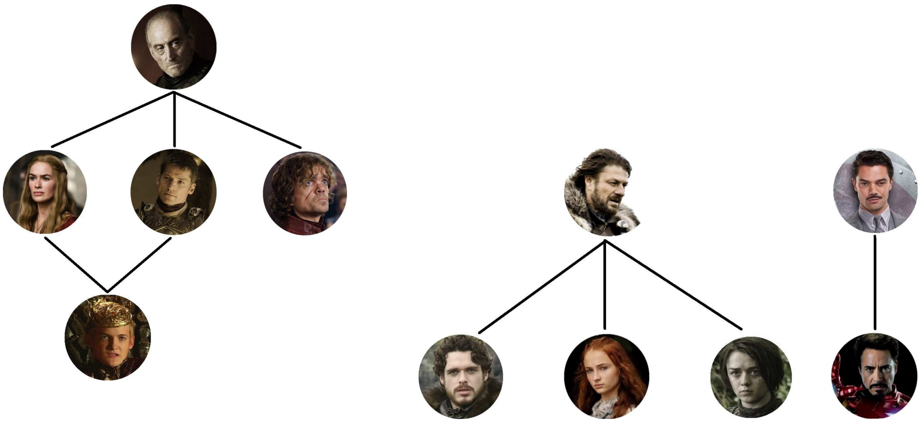

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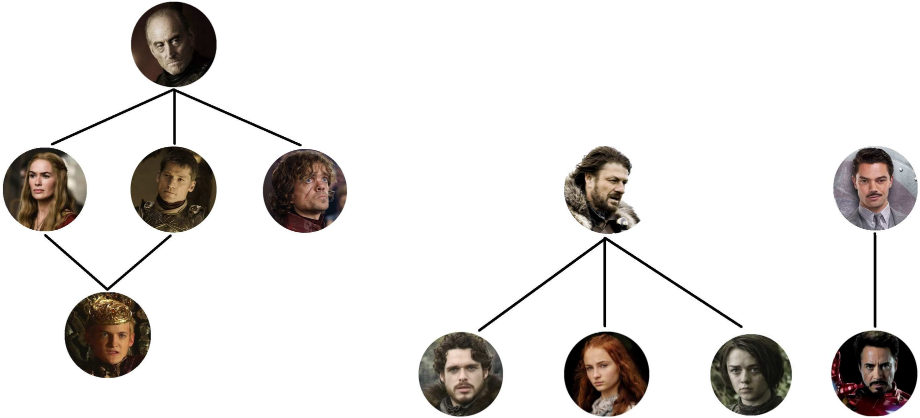

Theorem [L.N. Tolstoy (1878)]

Tree families are all alike ; every untree family is untree in its own way.

A factorial and biextendable set S is called a *tree set* of *characteristic* c if for any nonempty $w \in S$, the graph $E(w)$ is a tree and if $E(\varepsilon)$ is a union of c trees.

Example

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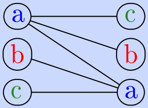
Proposition [Berthé, De Felice, **D.**, Leroy, Perrin, Reutenauer, Rindone (2014, 2015)]

Factors of an Arnoux-Rauzy word and regular interval exchange sets are both (uniformly) recurrent tree sets of characteristic 1.

Example (Tribonacci)

$E(\varepsilon)$



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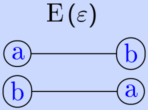
Let $A = \{a, b\}$ and θ be the identity on A. The set of factors of $(ab)^\omega$ is a specular set.



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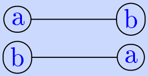


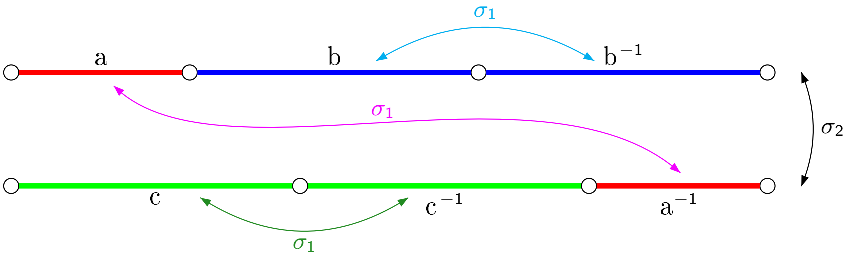
Proposition [*using* J. Cassaigne (1997)]

The factor complexity of a specular set is given by $p_0 = 1$ and $p_n = n(\mathrm{Card}(A) - 2) + 2$.

Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

The natural coding of a linear involution without connections is a specular set.

$$T = \sigma_2 \circ \sigma_1$$



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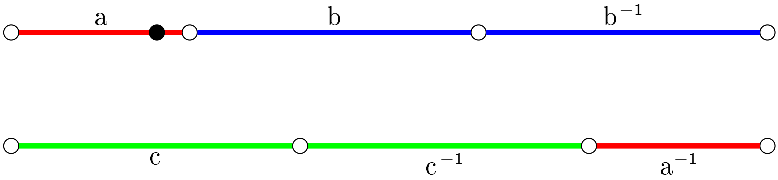

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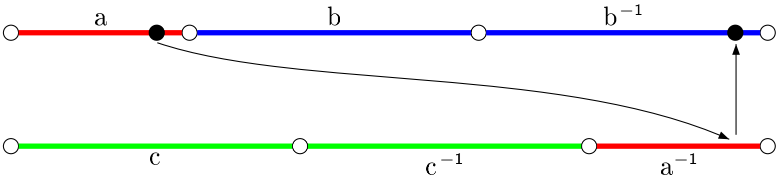

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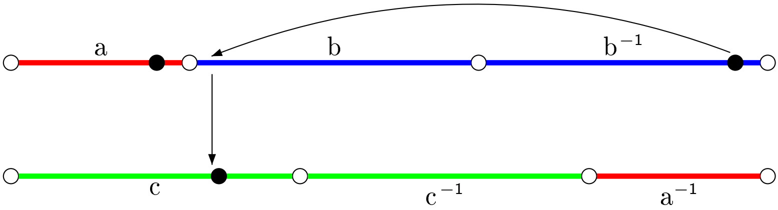

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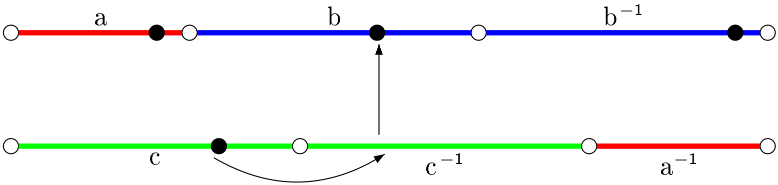

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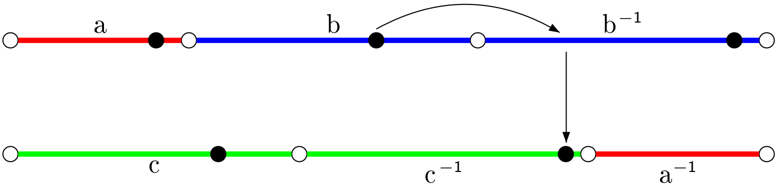

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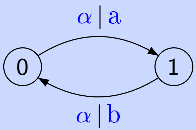
A *doubling transducer* is a transducer with set of states $\{0, 1\}$ such that :

1. the input automaton is a group automaton,
2. the output labels of the edges are all distinct.

Example

$\Sigma = \{\alpha\}$
$A = \{a, b\}$



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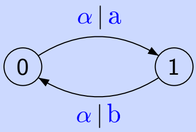

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The *image* of a set T is $\delta(T) = \delta_0(T) \cup \delta_1(T)$.

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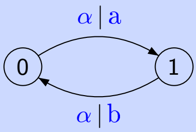

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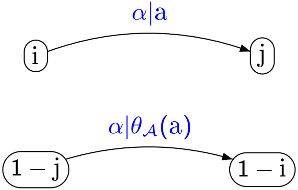
Proposition [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

The image of a tree set of characteristic 1 closed under reversal is a specular set with respect to $\theta_{\mathcal{A}}$.



Example (two doublings of Fibonacci on $\Sigma = \{\alpha, \beta\}$)

- Fac (abaababa $\cdots$) $\cup$ Fac (cdccdcdc $\cdots$)

$$\theta_{\mathcal{A}} : \begin{cases} a \mapsto c \\ b \mapsto d \\ c \mapsto a \\ d \mapsto b \end{cases}$$

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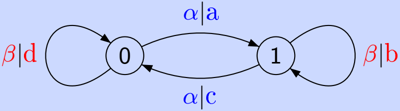

A letter is *even* if its two occurences (as a element of $\mathrm{L}(\varepsilon)$ and of $\mathrm{R}(\varepsilon)$) appear in the same tree of $\mathrm{E}(\varepsilon)$. Otherwise it is *odd*.

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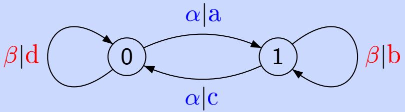

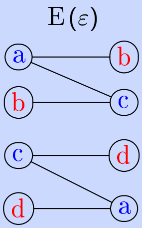

The letters b and d are even, while a and c are odd.

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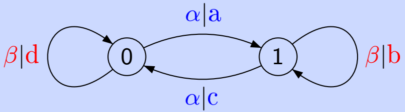

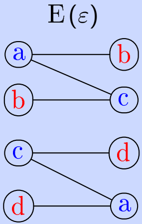

The letters b and d are even, while a and c are odd.

A word is *even* if it has an even number of odd letters. Otherwise it is *odd*.

A *right return word* to w in S is a nonempty word u such that $wu \in S \cap A^*w$, but has no internal factor equal to w.

We denote by $\mathcal{R}(w)$ the set of right return words to w in S.

Example (Fibonacci)

$\mathcal{R}(aa) = \{baa, babaa\}$.

$$\varphi(a)^\omega = abaababaabaababaababaabaababaabaab\cdots$$

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Proposition [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

For every w in a specular set, every word in $\mathcal{R}(w)$ is even.

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Cardinality Theorem for right return words [BDDDLPRR (2015)]

For any w in a recurrent specular set, $\mathrm{Card}\,(\mathcal{R}(w)) = \mathrm{Card}\,(A) - 1$.

A set $X \subset A^+$ of nonempty words over an alphabet A is a *bifix code* if it does not contain any proper prefix or suffix of its elements.

Example

- $\{aa, ab, ba\}$
- $\{aa, ab, bba, bbb\}$
- $\{ac, bcc, bcbca\}$
- {bagnole, char, chariotte}
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The *kernel* of a bifix code is the set of words of X which are internal factors of X.

Example

The kernel of the code $\{a, b, ba, aba\}$ is the set $\{b\}$.

A *complete return word* to a set $X \subset S$ is a word in the set $(S \cap XA^+ \cap A^+X) \setminus A^+XA^+$.

We denote by $\mathcal{CR}(X)$ the set of complete return words to X.

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Cardinality Theorem for complete return words [BDDDLPRR (2015)]

Let S be a recurrent specular set and $X \subset S$ be a finite bifix code with empty kernel. Then, $\mathrm{Card}(\mathcal{CR}(X)) = \mathrm{Card}(X) + \mathrm{Card}(A) - 2$.

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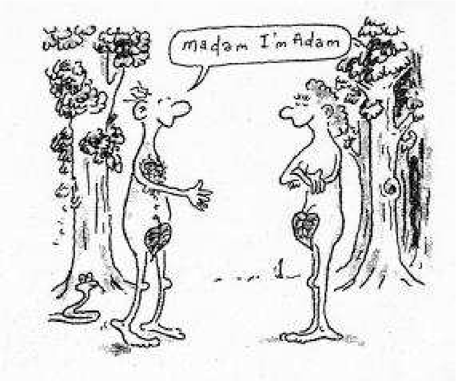

Theorem [X. Droubay, J. Justin, G. Pirillo (2001)]

A word of length n has at most $n+1$ palindrome factors.

A word with maximal number of palindromes is *full* (or *rich*). A factorial set is *full* if all its elements are full.

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Theorem [A. Glen, J. Justin, S. Widmer, L.Q. Zamboni (2009)]

A recurrent set closed under reversal is full iff every complete return word to a palindrome is a palindrome.

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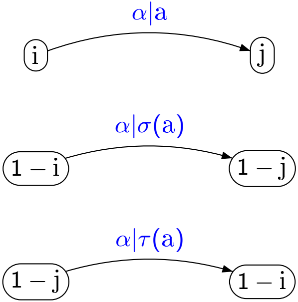
Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2016)]

Let T be a recurrent tree set of characteristic 1 closed under reversal.
The image of T under a doubling map $\mathcal{A}$ is $\mathrm{G}_{\mathcal{A}}$-full.

$$\mathrm{G}_{\mathcal{A}} = \{\mathrm{id}, \sigma, \tau, \sigma\tau\} \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$$

with σ an antimorphism and τ a morphism.



Theorem [E. Pelantová, Š. Starosta (2014)]

A set S closed under G is G-full if for every $w \in S$, every complete return word to the G-orbit of w is fixed by a nontrivial element of G.

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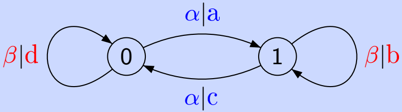
Example (doubling of Fibonacci)



$\sigma : \quad b \leftrightarrow d$

$\tau : \quad a \leftrightarrow c$, $\quad b \leftrightarrow d$

$\sigma\tau = \tau\sigma : \quad a \leftrightarrow c$

The $G_{\mathcal{A}}$-orbit of a is $\{a, c\}$.

All elements of $\mathcal{CR}(\{a, c\}) = \{abc, ac, ca, cda\}$ are fixed by $\sigma\tau \in G_{\mathcal{A}}$.

Conclusions

Summing up

- Tree and specular sets.
- Connections to specular groups.
- Doubling maps.
- Cardinality Theorems for return words.
- Palindromes and (G-)full sets.

Further Research Directions

and other works in progress

- Decidability of the tree (and specular) condition.
- Tree set and free groups

  Tree set of $\chi = 1 \Longrightarrow \mathcal{R}(\mathrm{w})$ is a basis of the free group for every w
- Explicit formula for number of G-palindromes.
- Generalization towards larger classes of groups (virtually free)


Merci
Thank You