Regular interval exchange sets over a quadratic field

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Words and morphisms

A morphism $f : A^* \to A^*$ is *primitive* if there exists $k > 0$ s.t. a appears in $f^k(b)$ for every a, b.

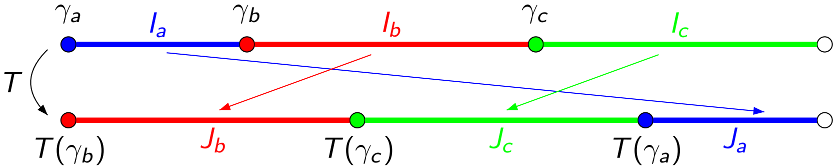
An infinite word y is *morphic* if $y = \sigma(x)$ with σ a morphism and $x = f^\omega(a)$ a fixpoint of a morphism f. If $\sigma = id$. the word is *purely morphic*. If f is primitive, the word is *primitive morphic*.

The set of factors $F(x)$ of an infinite word x is said (*purely*, *primitive*) *morphic* if x is (purely, primitive) morphic.

Interval exchange transformations

Let $(I_a)_{a\in A}$ be an ordered partition of $[\ell, r[$. An *interval exchange transformation* (IET) is a map $T : [\ell, r[\to [\ell, r[$ defined by

$$T(z) = z + \alpha_z \quad \text{if } z \in I_a.$$



Remark : The restriction of a coherent orientable linear involution to one of the components is a IET[1].

1. See Valérie Berthé's talk just next.

Regular interval exchange transformations

T is said to be *minimal* if for any $z \in [\ell, r[$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[\ell, r[$.

T is said *regular* if the orbits of the separation points $\neq \ell$ are infinite and disjoint.

Theorem [Keane, 1975]

A regular interval exchange transformation is minimal.

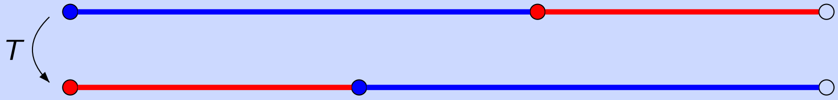
Natural coding

Let T be an IET relative to $(I_a)_{a \in A}$. The *natural coding* of T relative to $z \in [\ell, r[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in A^\omega$ defined by

$$a_n = a \quad \text{si } T^n(z) \in I_a.$$

Example

The *Fibonacci word* is the natural coding of the rotation of angle $\alpha = (3-\sqrt{5})/2$ relative to the point α, i.e. $T(z) = z + \alpha \bmod 1$.



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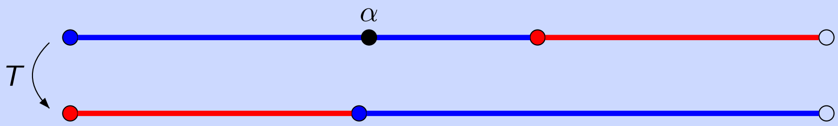

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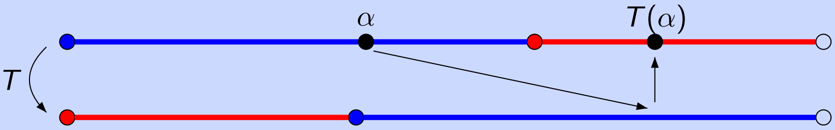

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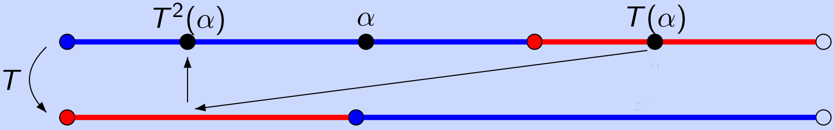

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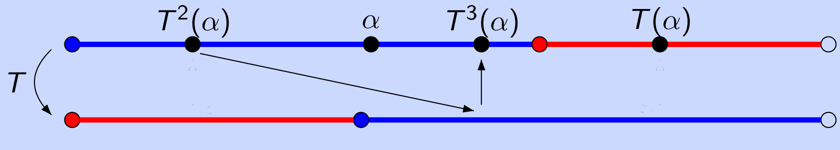

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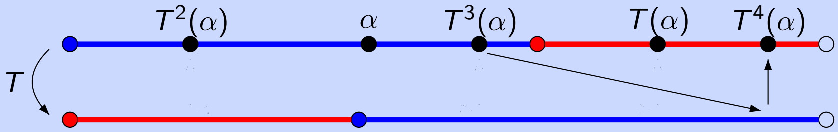

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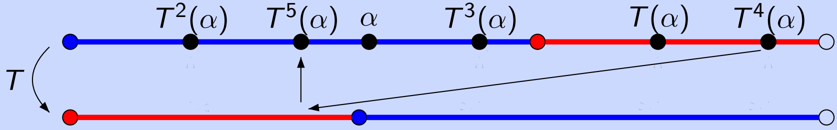


$$\Sigma_T(\alpha) = a\, b\, a\, a\, b\, a \cdots$$

Regular interval exchange sets

Proposition

If T is minimal, $F(\Sigma_T(z))$ does not depend on z.

When T is regular (minimal), $F(T) = F(\Sigma_T(z))$ is said a *regular* (*minimal*) *interval exchange set*.

Example

The *Fibonacci set* is the set of factors of a natural coding of the rotation of angle $\alpha = (3 - \sqrt{5})/2$.

$$F(T) = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, \dots\}$$

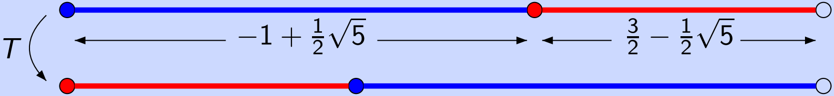
Regular interval exchange sets over a quadratic field

Theorem

Let T be a regular IET defined over a quadratic field. Then the interval exchange set $F(T)$ is primitive morphic.

Example

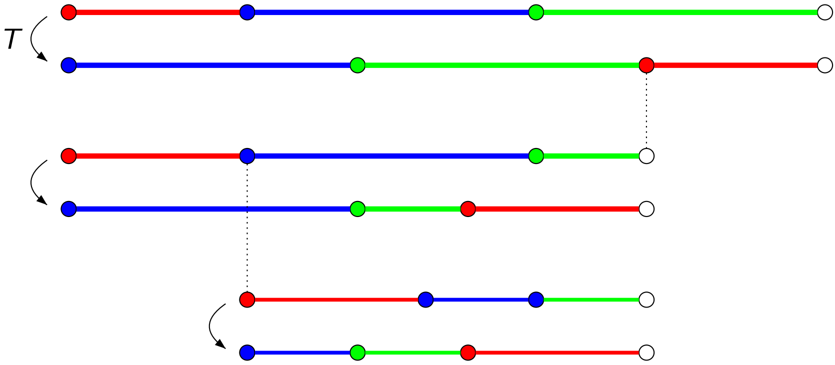
$$|I_a|, |I_b| \in \mathbb{Q}[\sqrt{5}]$$



$$F(T) = F(x) \qquad \text{with } x = id. \circ f^{\omega}(a)$$

$$f : \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$

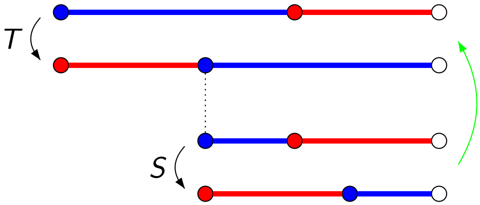
Two-sided Rauzy induction



Theorem

T regular IET $\Longrightarrow$ induced trasformations still regular IETs.

Rauzy induction and natural coding



$$\theta = \begin{cases} a \mapsto a \\ b \mapsto ba \end{cases}$$

$$\Sigma_T(\alpha) = a\,\underline{b\,a}\,a\,\underline{b\,a}\,\underline{b\,a}\,a\,\underline{b\,a}\,a\,\underline{b\,a}\,\underline{b\,a}\cdots$$

$$\Sigma_S(\alpha) = a\,b\,a\,b\,b\,a\,b\,a\,b\,b\cdots$$

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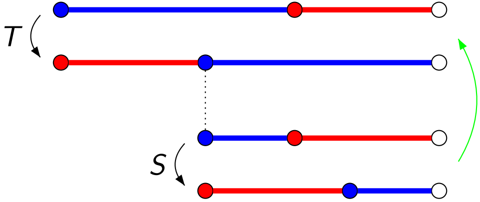

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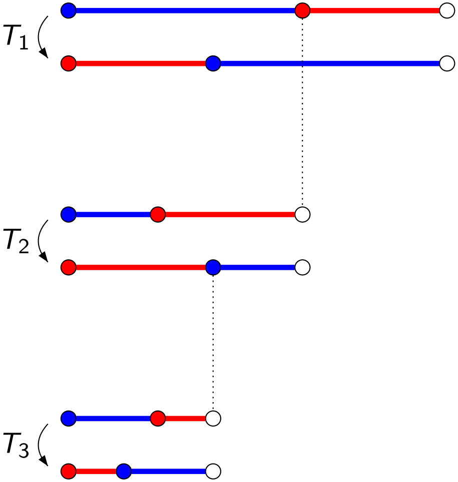
$$\Sigma_S(\alpha) = a\, b\, a\, b\, b\, a\, b\, a\, b\, b \cdots$$

Theorem

Let T be a regular IET and S a IET obtained by Rauzy induction. There exists an automorphism θ of the free group A° s.t. $\Sigma_T(z) = \theta\left(\Sigma_S(z)\right)$ for every $z \in D(S)$.

Equivalent IETs



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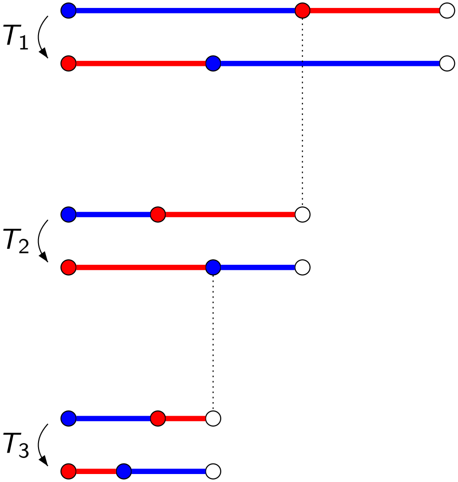

$$\frac{|I_a|}{|I_b|} = \phi = \frac{1+\sqrt{5}}{2} = 1.61803\ldots$$

$$= 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}$$

$$= [1; 1, 1, 1, \ldots]$$

$$\frac{|I_a|}{|I_b|} = \frac{1}{\phi}$$

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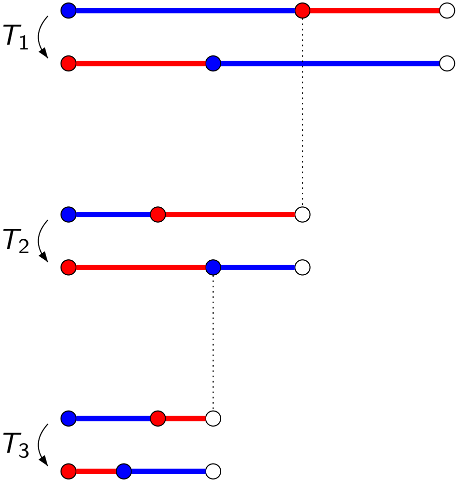

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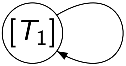

Rauzy induction over a quadratic field

Theorem (Lagrange, 1770)

Quadratic irrationals have a periodic continued fraction expansion.

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Starting from a regular IET and inducing over a particular kind of subinterval, we obtain a finite number of distinct equivalence classes.

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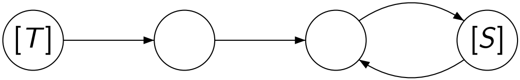
Theorem (D., 2014)

Starting from a regular IET we obtain, by two-sided Rauzy induction, a finite number of distinct equivalence classes.

Rauzy induction over a quadratic field

$$\frac{|I_a|}{|I_b|} = \frac{4+\sqrt{2}}{2} = 2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cdots}}} = [2; 1, 2, 2, 2, \cdots]$$



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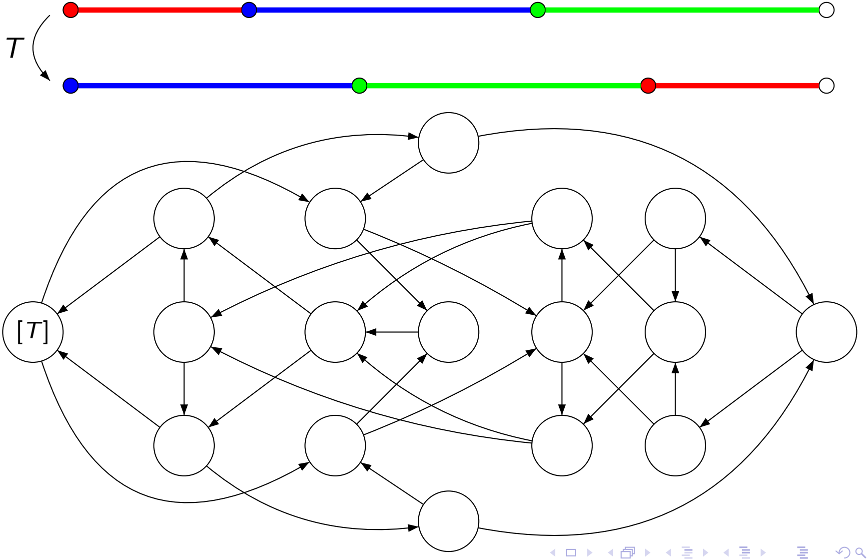
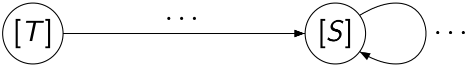


Rauzy induction over a quadratic field

Corollary

Let T be a regular IET defined over a quadratic field. In the equivalence graph we can find a path from $[T]$ to a node $[S]$ followed by a cycle.



Regular interval exchange sets over a quadratic field

Theorem

Let T be a regular IET defined over a quadratic field. Then the interval exchange set $F(T)$ is [primitive] morphic.

Proof.

There exists a point $z \in D(S)$ and two automorphisms θ, η of the free group s.t.

$$\Sigma_T(z) = \theta(\Sigma_S(z))$$

with $\Sigma_S(z)$ fixpoint of η.

Questions