Clustering of return words in languages of Interval Exchange Transformations

Francesco Dolce and Christian B. Hughes

WORDS 2025

Nancy, 2 juillet 2025

louche – chelou

louche – chelou

fou – ouf

louche – chelou

fou – ouf

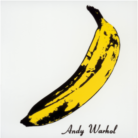
Two words uv and vu are *conjugates*.

$[w] = \{w' = uv \mid w = vu\}$

The smallest element in $[w]$ (w primitive) is a *Lyndon word*.

Burrows-Wheeler Transform

Let $(\mathcal{A}, <)$ be an ordered alphabet and $w \in \mathcal{A}^*$.



`b a n a n a`

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```
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  a n a n a b
    n a n a b a
      a n a b a n
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a	n	a	n	a	b
b	a	n	a	n	a
n	a	b	a	n	a
n	a	n	a	b	a

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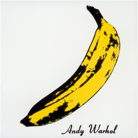

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$$\mathrm{bwt}_{\mathcal{A}}(\mathtt{banana}) = \mathtt{nnbaaa}$$

Proposition

w is a conjugate of u^p if and only if $\mathrm{bwt}_{\mathcal{A}}(u) = b_1 \cdots b_n$ and $\mathrm{bwt}_{\mathcal{A}}(w) = b_1^p \cdots b_n^p$.

$\mathrm{bwt}_{\mathcal{A}}(\mathtt{bon}) = \mathtt{nob}$ $\mathrm{bwt}_{\mathcal{A}}(\mathtt{bonbon}) = \mathtt{nnoobb}$

Clustering words

A word $w \in \mathcal{A}^*$ is π-*clustering* for $\mathcal{A}$ if $\mathrm{bwt}_{\mathcal{A}}(w) = a_{\pi(1)}^{k_1} \cdots a_{\pi(d)}^{k_d}$, where $k_i = |w|_{a_{\pi(i)}}$.

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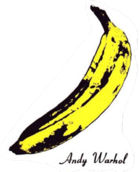
$\mathcal{A} = \{\mathtt{a} < \mathtt{b} < \mathtt{n}\}$,

$\mathrm{bwt}_{\mathcal{A}}(\mathrm{banana}) = \mathtt{nnbaaa}$,

$$\pi = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{n} \\ \mathtt{n} & \mathtt{b} & \mathtt{a} \end{pmatrix}$$



$\mathcal{A}' = \{\mathtt{a} < \mathtt{n} < \mathtt{b}\}$,

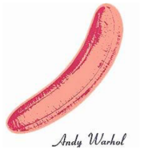
$\mathrm{bwt}_{\mathcal{A}'}(\mathrm{banana}) = \mathtt{bnnaaa}$,

$$\pi' = \begin{pmatrix} \mathtt{a} & \mathtt{n} & \mathtt{b} \\ \mathtt{b} & \mathtt{n} & \mathtt{a} \end{pmatrix}$$

$\mathcal{A}'' = \{\mathtt{n} < \mathtt{a} < \mathtt{b}\}$

$\mathrm{bwt}_{\mathcal{A}''}(\mathrm{banana}) = \mathtt{aabnna}$

not clustering



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Proposition

Let $w = u^p \in \mathcal{A}^*$. Then w is π-clustering for $\mathcal{A}$ if and only if u is π-clustering for $\mathcal{A}$.

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Theorem [Mantaci, Restivo, Sciortino (2003)]

Over a binary alphabet (perfectly) clustering words are exactly powers of *Christoffel words* (i.e., *finite standard Sturmian words*) and their conjugates.

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And over larger alphabets ?

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Clustering words

A characterization with palindromes

Theorem [Lapointe, Reutenauer (2024)]

Let $w \in \mathcal{A}^*$ be a primitive word. The following are equivalent :

- w is a perfectly clustering Lyndon word
- $\begin{cases} w = a_1 p_1 a_2 p_2 \cdots p_{d-1} a_d, & p_i = \tilde{p}_i \quad (\textit{special factorization}) \\ w = q_1 q_2, & q_j = \tilde{q}_j \end{cases}$
- $\begin{cases} w = a_1 p_1 a_2 p_2 \cdots p_{d-1} a_d \\ [w] = [a_d p_{d-1} \cdots p_2 a_2 p_1 a_1] \end{cases}$

$\mathrm{bwt}_{\mathcal{A}}(\mathtt{ananas}) = \mathtt{n}^2 \mathtt{s}^1 \mathtt{a}^3$

$\mathtt{a} \cdot \varepsilon \cdot \mathtt{n} \cdot \mathtt{ana} \cdot \mathtt{s}$

$\mathtt{anana} \cdot \mathtt{s}$

$[\mathtt{a}\,\varepsilon\,\mathtt{n}\,\mathtt{ana}\,\mathtt{s}] = [\mathtt{s}\,\mathtt{ana}\,\mathtt{n}\,\varepsilon\,\mathtt{a}]$

Clustering words and (D)IETs

Theorem [Ferenczi, Zamboni (2013)]

Let $w \in \mathcal{A}^*$ be primitive, with $\mathrm{Card}(\mathcal{A}) = d$. The following are equivalent :

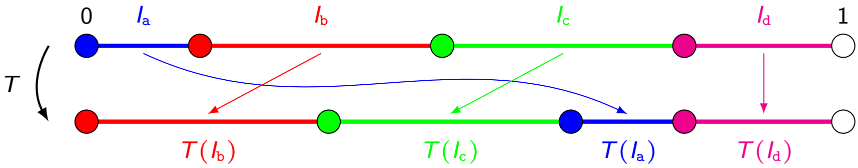
1. w is π-clustering ;
2. ww occurs in the trajectory of a *minimal* d-*DIET* with permutation π ;
3. ww occurs in the trajectory of a d-*DIET* with permutation π ;
4. ww occurs in the trajectory of a d-*IET* with permutation π.

IETs ? DIETs ?

Interval exchanges

Let $(I_a)_{a\in\mathcal{A}}$ be a partition of $[\ell, r)$.
An *interval exchange transformation* (IET) is a map $T : [\ell, r) \to [\ell, r)$ defined by

$$T(z) = z + \tau_a \quad \text{if} \quad z \in I_a.$$



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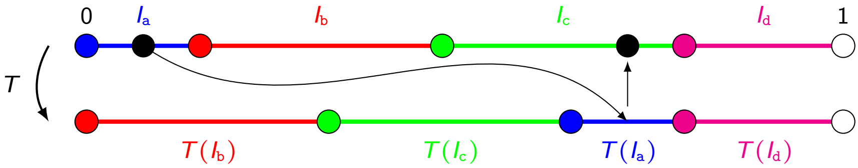

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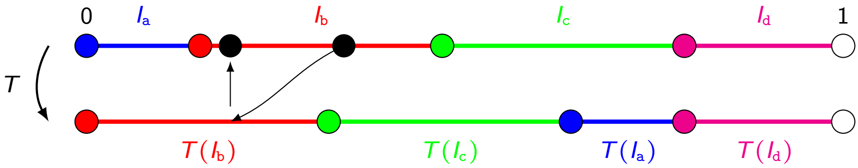

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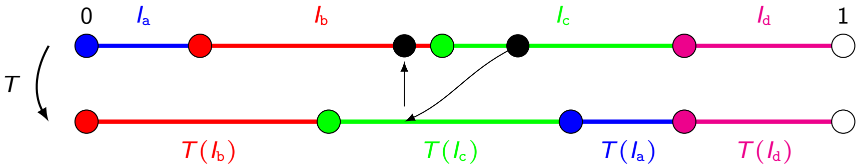

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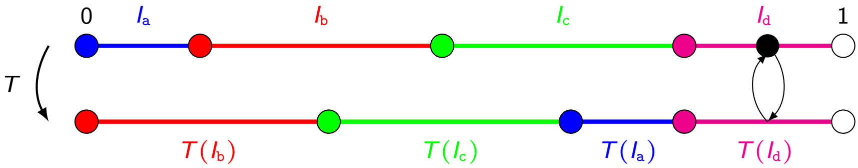

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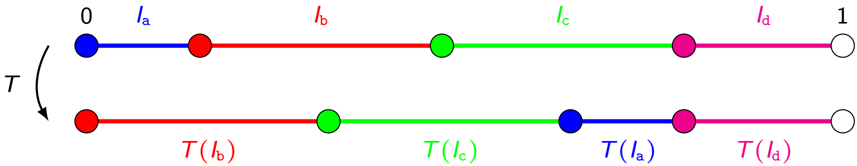


$$\mathcal{A} = \{\mathtt{a} < \mathtt{b} < \mathtt{c} < \mathtt{d}\} \qquad \pi = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} \\ \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{d} \end{pmatrix}$$

Minimality and regularity

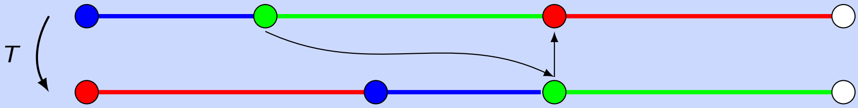
T is *minimal* if for any $z \in [\ell, r)$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[\ell, r)$.

T is *regular* if the orbits of the non-zero formal discontinuities are infinite and disjoint.

Theorem [Keane (1975)]

A regular interval exchange transformation is minimal.

Example (the converse is not true)

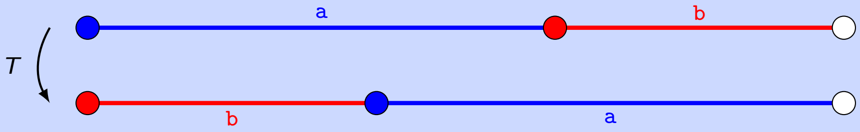


Trajectories

The *trajectory* of $z \in [\ell, r)$ under T is the infinite word $\Omega_T(z) = w_0 w_1 \cdots \in \mathcal{A}^\omega$ defined by

$$w_n = a \qquad \text{if } T^n(z) \in I_a.$$

Example (Fibonacci)



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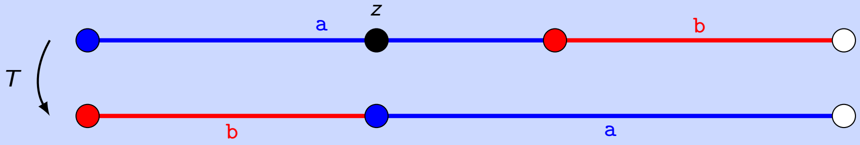

$\Omega_T(z) = \mathtt{a}$

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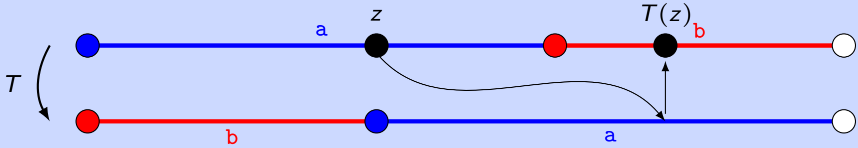

$\Omega_T(z) = ab$

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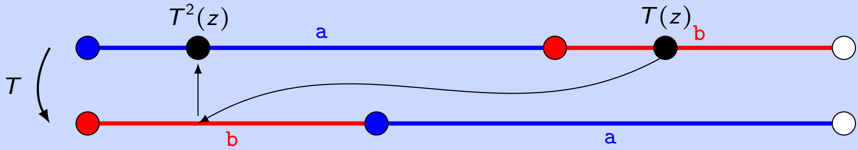

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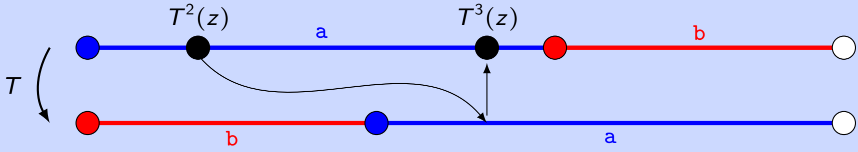

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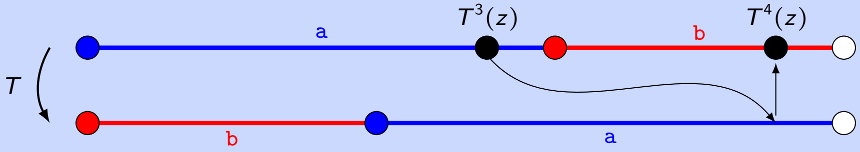

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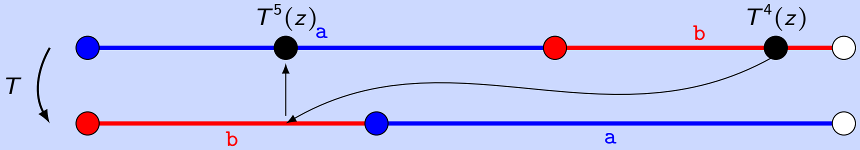

$$\Omega_T(z) = \mathtt{abaaba}\cdots$$

Interval exchange languages

The set $\mathcal{L}(T) = \bigcup_{z \in [\ell, r)} \mathcal{L}(\Omega_T(z))$ is a (*minimal*, *regular*) *interval exchange language*.

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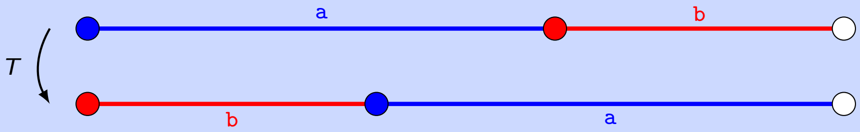

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Remark. When T is minimal, $\mathcal{L}(\Omega_T(z))$ does not depend on the point z.

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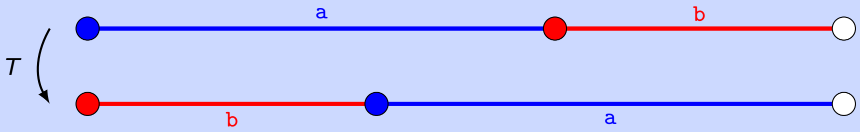
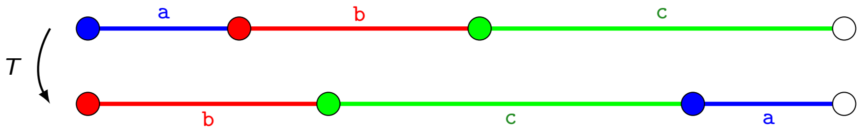


$$\mathcal{L}(T) = \{\varepsilon, \mathtt{a}, \mathtt{b}, \mathtt{aa}, \mathtt{ab}, \mathtt{ba}, \mathtt{aab}, \mathtt{aba}, \mathtt{baa}, \mathtt{bab}, \mathtt{aaba}, \ldots\}$$

From letters to words

Given a IET T and a word $w = w_0 w_1 \cdots w_m \in \mathcal{A}^*$, let

$$I_w = I_{w_0} \cap T^{-1}(I_{w_1}) \cap \cdots \cap T^{-m}(I_{w_m}) \qquad \text{(by convention } I_\varepsilon = [\ell, r))$$



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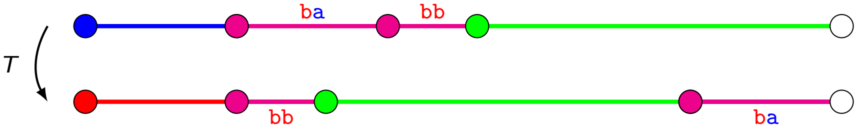


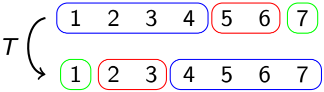
$$I_{ba} = I_b \cap T^{-1}(I_a) \qquad T^2(I_{ba}) = T^2(I_b) \cap T(I_a)$$

Thus $\Omega_T(z)$ starts with w for every $z \in I_w$.

From IETs to DIETs

Let $(n_1, n_2, \dots, n_d)$ be a composition of $n = \sum n_i$.
A *discrete interval exchange transformation* (DIET) is a map $T : \mathbb{N}_n \to \mathbb{N}_n$ defined by

$$T(k) = k + t_i \quad \text{if} \quad k \in \left[\sum_{j<i} n_j, \sum_{j \le i} n_j \right].$$



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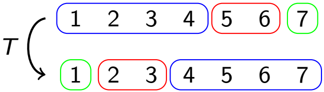

$$\Omega_T(1) = 1\,4\,7\,1\,4\,7\,1\,4\,7 \cdots = (1\,4\,7)^\omega$$

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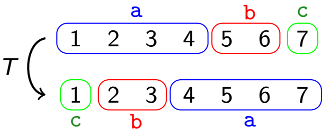

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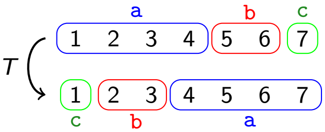

$$\Omega_T(1) = \mathtt{a\ a\ c\ a\ a\ c\ a\ a\ c} \cdots = (\mathtt{a\ a\ c})^\omega$$

$$\mathcal{L}(T) = \mathcal{L}\left((\mathtt{aac})^\omega \cup (\mathtt{ab})^\omega \cup (\mathtt{ab})^\omega\right)$$

Return words

Question [Lapointe (2021)]

Are return words of a symmetric IET perfectly clustering words ?

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$$\mathcal{R}_{\mathcal{L}}(u) = \{w \in \mathcal{A}^* \mid uw \in (\mathcal{L} \cap \mathcal{A}^* u) \setminus \mathcal{A}^+ u \mathcal{A}^+\}$$

Example (Fibonacci)

$$\mathbf{f} = \underline{\mathtt{aba}}\,\underline{\mathtt{aba}}\,\mathtt{baabaa}\,\underline{\mathtt{aba}}\,\underline{\mathtt{ba}}\,\mathtt{abab}\,\mathtt{aaba}\cdots$$

$\mathcal{R}(\mathtt{aba}) = \{\mathtt{aba},\ \mathtt{ba}\}$

Induced transformations

Let T be a minimal IET and $J \subset [\ell, r)$.
The *transformation induced* by T (*first return map* of T) on J is $T' : J \to J$ defined by

$$T'(z) = T^{\nu(z)}(z) \qquad \text{with} \quad \nu(z) = \min\{n > 0 \mid T^n(z) \in J\}$$

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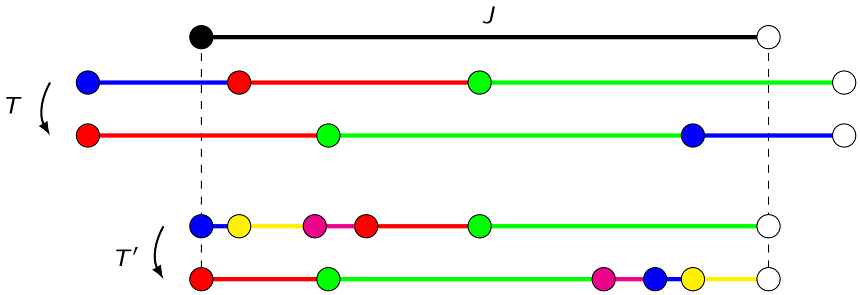
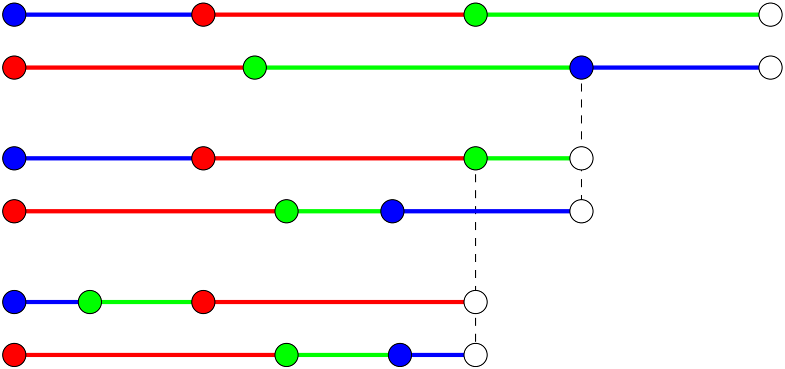
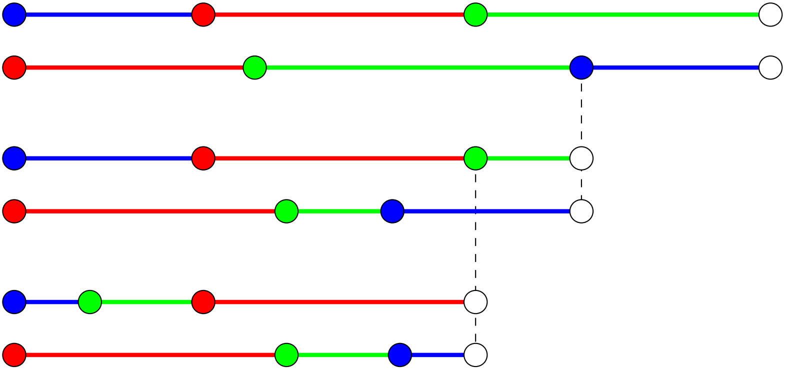


Right Rauzy induction



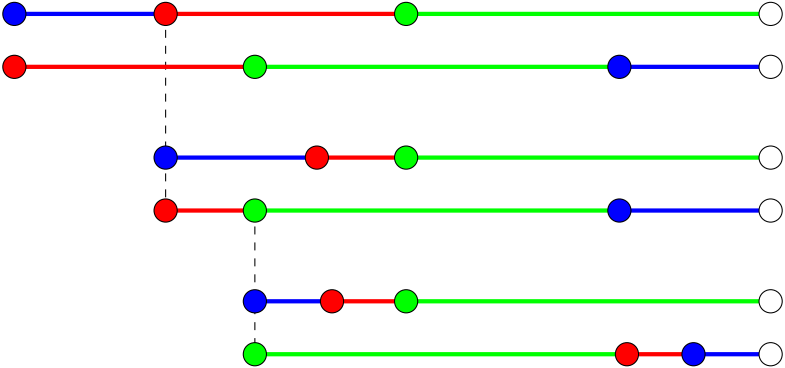
Right Rauzy induction



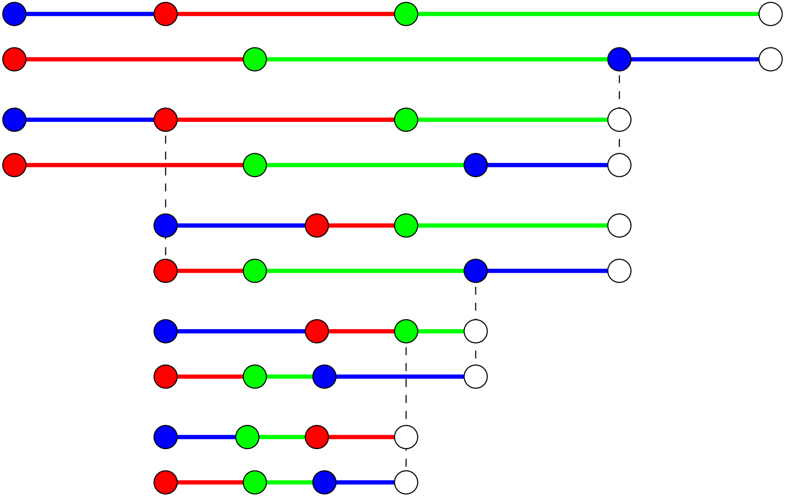
Theorem [Rauzy (1979)]

If T is regular, the right Rauzy induced transformation is regular on the same alphabet.

Left Rauzy induction



Two-sided Rauzy induction



Rauzy induction on I_w

Theorem [D., Perrin (2017)]

Let T be a regular IET and $w \in \mathcal{L}(T)$.
The regular IET T' induced on I_w can be obtained by two-sided Rauzy induction.

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Moreover, each step is associated with an automorphism of $F_{\mathcal{A}}$ of the form

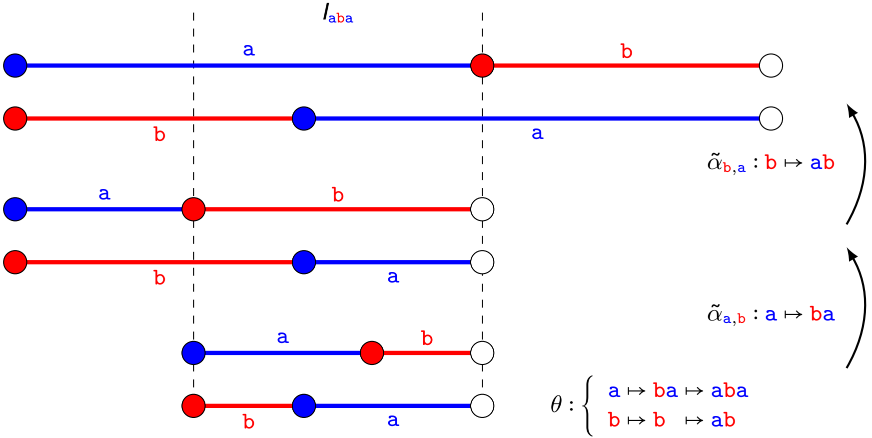
$$\alpha_{a,b} = \begin{cases} a \mapsto ab \\ c \mapsto c \quad \forall\, c \neq a \end{cases} \qquad \text{or} \qquad \tilde{\alpha}_{a,b} = \begin{cases} a \mapsto ba \\ c \mapsto c \quad \forall\, c \neq a \end{cases}$$

and for every $z \in I_w$

$$\Omega_T(z) = \theta\left(\Omega_{T'}(z)\right).$$

(θ is obtained from the morphisms above, but backwards!)

Rauzy induction on I_w

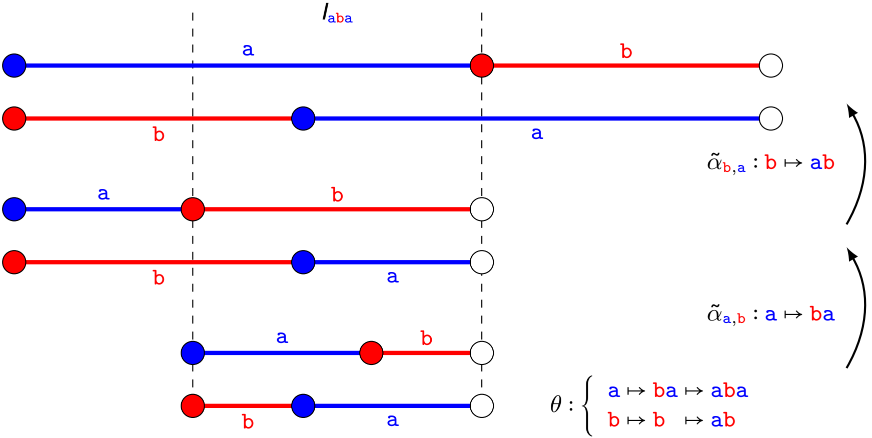


$\tilde{\alpha}_{b,a} : b \mapsto ab$

$\tilde{\alpha}_{a,b} : a \mapsto ba$

$$\theta : \begin{cases} a \mapsto ba \mapsto aba \\ b \mapsto b \mapsto ab \end{cases}$$

Rauzy induction on I_w



$$\theta : \begin{cases} a \mapsto ba \mapsto aba \\ b \mapsto b \quad \mapsto ab \end{cases}$$

$$\mathcal{R}(aba) = \{aba, ab\}$$

Return words in IETs are clustering

a.k.a. the main result

Corollary

$\mathcal{R}(w) = \theta(\mathcal{A})$.

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Proposition (way too technical to be stated properly)

Under the "right conditions" each $\alpha_{a,b}$ (resp., $\tilde{\alpha}_{a,b}$) preserves clustering.

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Under the "right conditions" each $\alpha_{a,b}$ (resp., $\tilde{\alpha}_{a,b}$) preserves clustering.

Theorem

Let T be regular IET and $u \in \mathcal{L}(T)$. Each $w \in \mathcal{R}(u)$ is clustering.

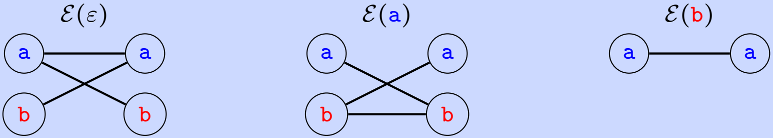
Extension graphs

The *extension graph* of $w \in \mathcal{L}$ is the bipartite graph $\mathcal{E}(w)$ with vertices

$$L(w) = \{a \in \mathcal{A} \mid aw \in \mathcal{L}\} \qquad \text{and} \qquad R(w) = \{b \in \mathcal{A} \mid wb \in \mathcal{L}\},$$

and edges $B(w) = \{(a,b) \in \mathcal{A} \times \mathcal{A} \mid awb \in \mathcal{L}\}$.

Example (Fibonacci, $\mathcal{L} = \{\varepsilon, \mathtt{a}, \mathtt{b}, \mathtt{aa}, \mathtt{ab}, \mathtt{ba}, \mathtt{aba}, \mathtt{baa}, \mathtt{bab}, \ldots\}$)



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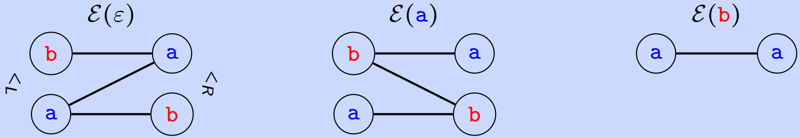
$\mathcal{E}(w)$ is *compatible* with two orders $<_L, <_R$ on $\mathcal{A}$ if for every $(a, b), (c, d) \in B(w)$

$$a <_L c \quad \Rightarrow \quad b \leq_R d.$$

Example (Fibonacci, $\mathtt{b} <_L \mathtt{a}$, and $\mathtt{a} <_R \mathtt{b}$)



See the forest for the IETs

It's all Greek to me!

$\mathcal{L}$ is *dendric* (δένδρον) if every $\mathcal{E}(w)$ is a tree. It is *alsinic* (ἄλσος) $\mathcal{E}(w)$ is a forest.

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Theorem [Ferenczi, Zamboni (2008) ; Ferenczi, Hubert, Zamboni (2024)]

- $\mathcal{L}$ is an interval exchange language iff it is (uniformly) recurrent ordered alsinic.
- $\mathcal{L}$ is a minimal interval exchange language iff it is aperiodic (uniformly) recurrent ordered alsinic.
- $\mathcal{L}$ is a regular interval exchange language iff it is (uniformly) recurrent ordered dendric.

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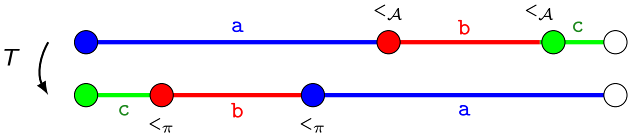
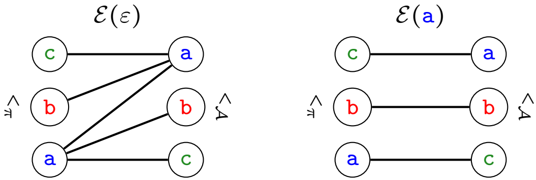
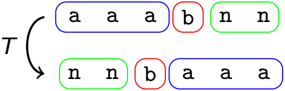


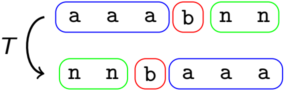


See the forest for the DIETs



$\mathrm{bwt}_{\{\mathtt{a}<\mathtt{b}<\mathtt{n}\}}(\mathtt{banana}) = \mathtt{nnbaaa}$

See the forest for the DIETs



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Theorem [Ferenczi, Hubert, Zamboni (2023)]

A word $w \in \mathcal{A}^*$ is π-clustering if and only if for every (bispecial) word $v \in \mathcal{L}(w^\omega)$ the graph $\mathcal{E}(v)$ is compatible with $<_\mathcal{A}$ and $<_\pi$.

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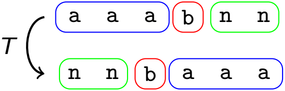

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Corollary

A word $w \in \mathcal{A}^*$ is π-clustering if and only if $\mathcal{L}(w^\omega)$ is ordered alsinic for $<_\mathcal{A}$ and $<_\pi$.

Where to go from here ?

- What about the permutation π ?

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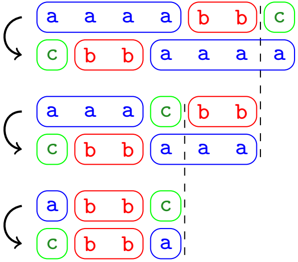

$$\mathcal{L}\left((\mathtt{aac})^\omega \cup (\mathtt{ab})^\omega \cup (\mathtt{ab})^\omega\right)$$

$$\theta : \begin{cases} \mathtt{a} \mapsto \mathtt{a} \mapsto \mathtt{a} \\ \mathtt{b} \mapsto \mathtt{ab} \mapsto \mathtt{ab} \\ \mathtt{c} \mapsto \mathtt{c} \mapsto \mathtt{ac} \end{cases}$$

$$\mathcal{R}(\mathtt{a}) = \{\mathtt{a}, \mathtt{ab}, \mathtt{ac}\}$$

Cimer coupbeau !