Column representation of episturmian words in cellular automata

Francesco Dolce

joint work with Pierre-Adrien Tahay

19e Journées Montoises d'informatique théorique

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Cellular automata

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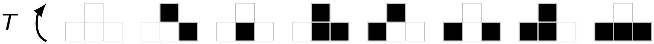

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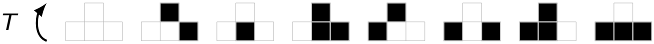

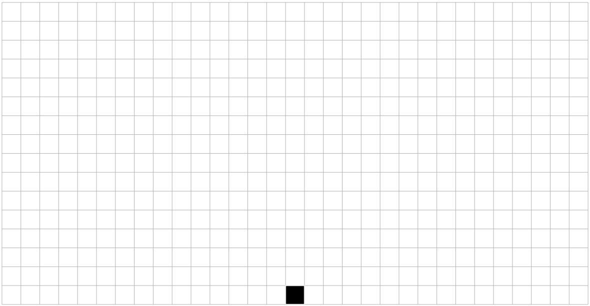

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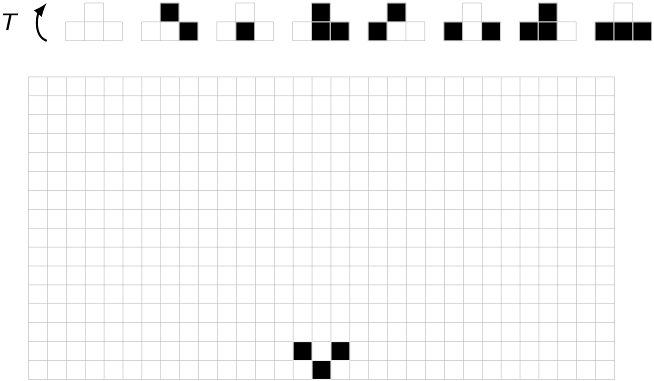

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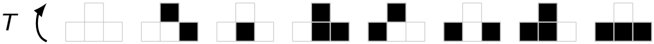

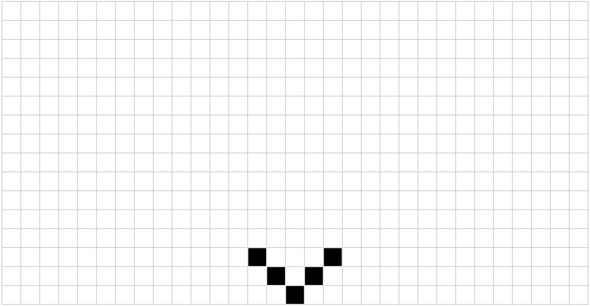

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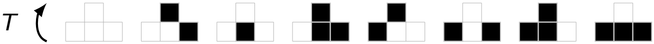

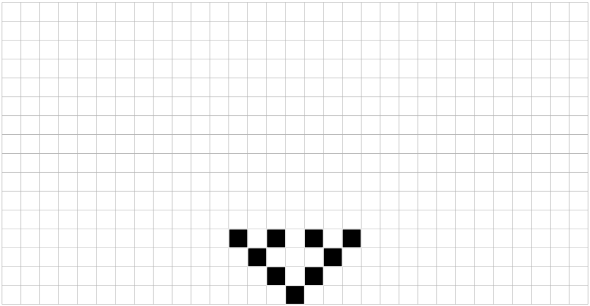

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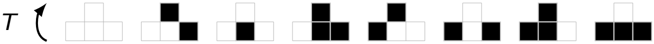

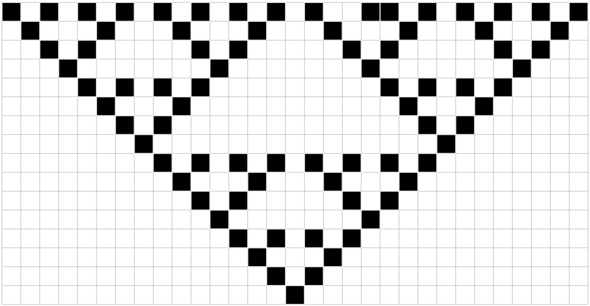

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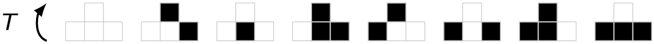
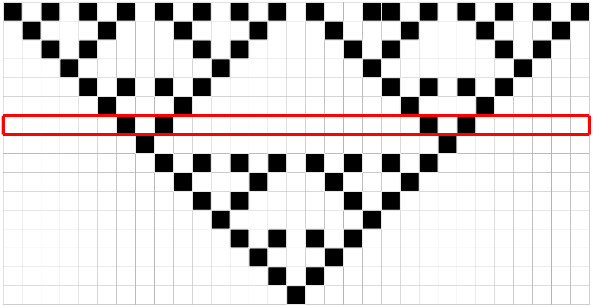
Elements of $\mathcal{A}^{\mathbb{Z}}$ are called *configurations*.
A configuration $\mathbf{x} = (\mathbf{x_k})_\mathbf{k}$ is *finite* if $\{k \ : \ x_k \neq 0\}$ is finite.





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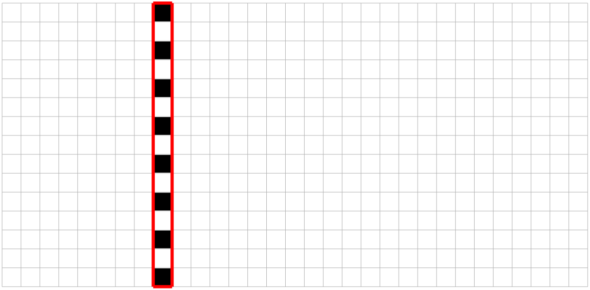

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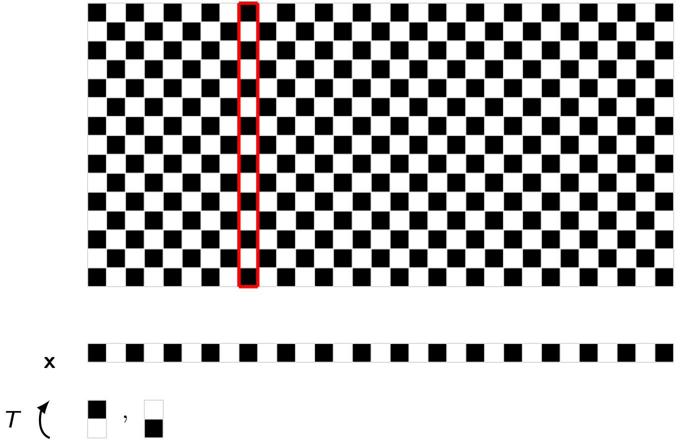


Column representation

Question : Which infinite sequence can be represented *using a finite number of states* on a column of a CA ?

Formally, given

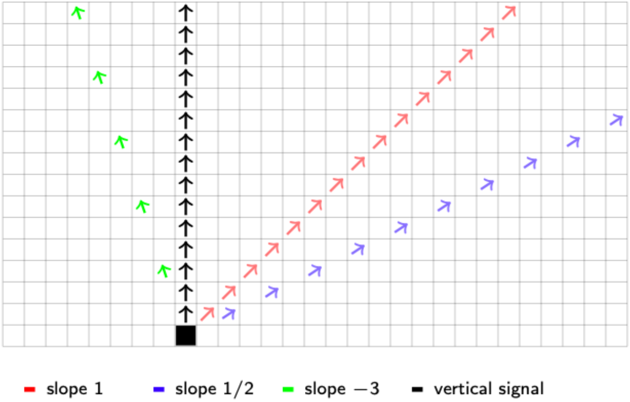
$$\mathcal{S} = \{ (T^n(\mathbf{x})_0)\, n \geq 0 \in \mathcal{A}^{\mathbb{N}} \ : \ T \text{ is a } 0\text{-quiescent CA on } \mathcal{A}^{\mathbb{Z}} \text{and } \mathbf{x} \text{ is finite} \}$$

determine whether $\mathbf{w} \in \mathcal{S}$ (and if possible construct the CA).

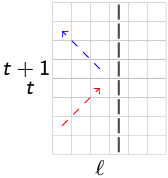
A cellular automaton T is 0-*quiescent* if $T(0^{\mathbb{Z}}) = \cdots 000 \cdots$.

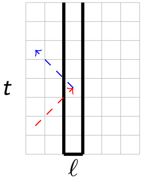
Signals

The *slope* of a signal is the ratio $\frac{t'-t}{\ell'-\ell}$.



Walls





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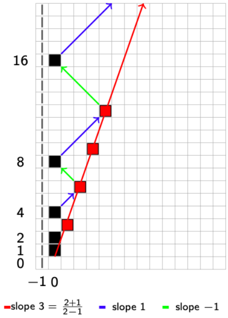

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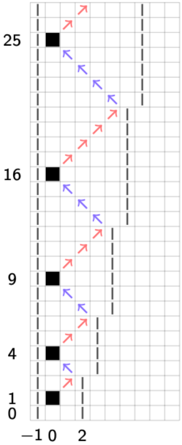

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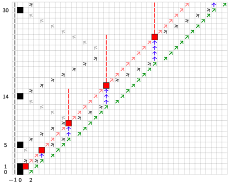

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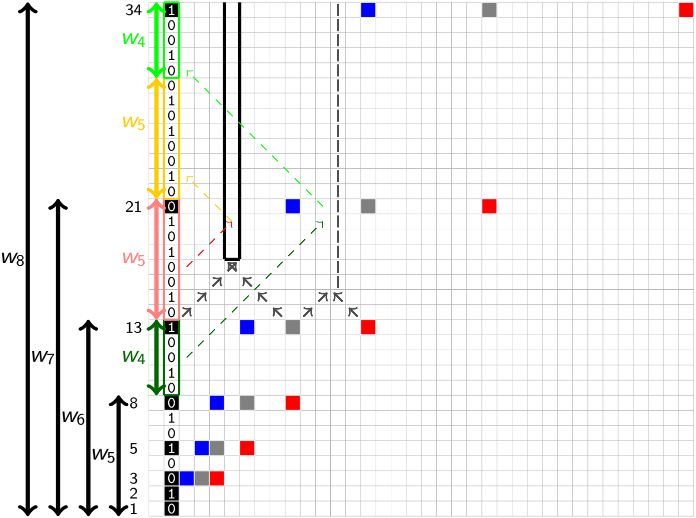
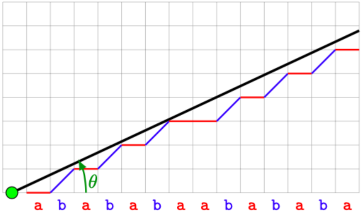
- **Sturmians with quadratic slope** [D., Tahay (2022)]
- **Episturmians, fixed points of a morphism** [D., Tahay (2024+)]

Sturmians : an example

Fibonacci



Sturmians : which prefixes ?



Theorem

The unique characteristic Sturmian word with slope and intercept α is $\lim_{n\to\infty} v_n$, where

$$v_{-1} = 1, \qquad v_0 = 0 \qquad \text{and} \qquad v_n = v_{n-1}^{d_n} v_{n-2} \qquad \text{for every} \quad n \geq 1$$

with
$$\alpha = [0, d_1 + 1, d_2, d_3, \ldots] = (d_1 + 1) + \cfrac{1}{d_2 + \cfrac{1}{d_3 + \ddots}}.$$

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Note : $u_n = w_{n-2}w_{n-3}\cdots w_0$

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The Last of runs

Let $\ell(k)$ be the last occurrence in Δ of the run of a_{k+1} before the k-th run.

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Characterizing these prefixes

Proposition [Peltomäki (2024)]

Let $\mathbf{w}$ be an epistandard word with directive sequence $\Delta = a_1^{e_1} a_2^{e_2} \cdots$. Then

- $w_k = \left(\prod_{i=1}^{k} w_{k-i}^{e_{k-i+1}}\right) a_{k+1}$ if a_{k+1} is <u>not</u> a factor of w_{k-1},
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  $w_3 = w_2^3 w_1^1 w_0^2 a_4$
  $= 0010\,0010\,0010 \cdot 001 \cdot 0\,0 \cdot 2$
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So, when can we construct the CA ?

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A strict epistandard word has a periodic directive sequence <u>if and only</u> it is the fixed point of a morphism.

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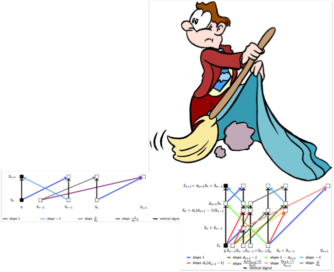

Merci