Eventually dendric shifts

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CZECH TECHNICAL UNIVERSITY IN PRAGUE

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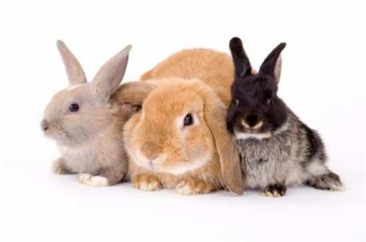
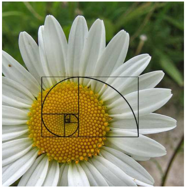
Fibonacci

$$\mathbf{x} = abaababaabaababa\cdots$$

$$\mathbf{x} = \lim_{n\to\infty} \varphi^n(a) \qquad \text{where} \qquad \varphi : \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$





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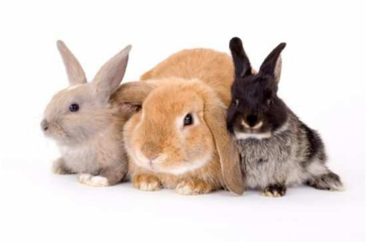

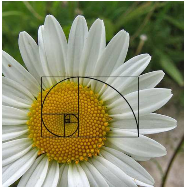

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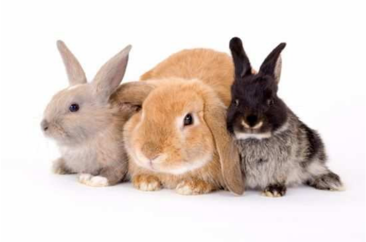

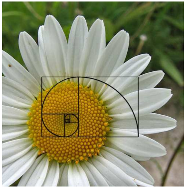

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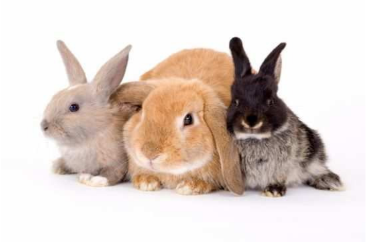

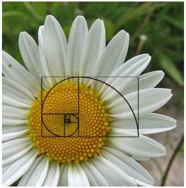

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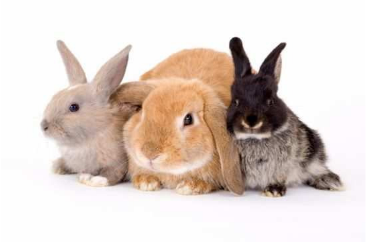

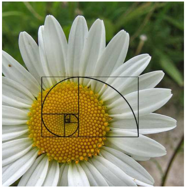

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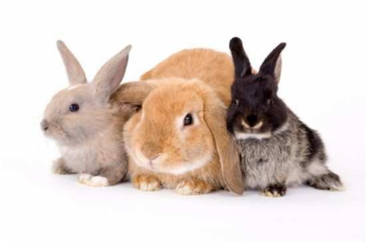

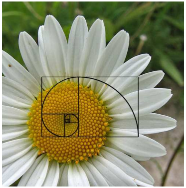

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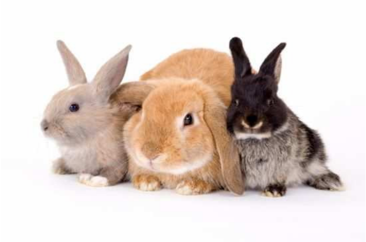
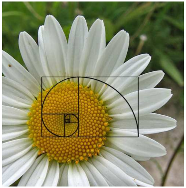




Fibonacci

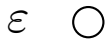
$$\mathbf{x} = abaababaabaababa \cdots$$

The *Fibonacci set* (set of factors of $\mathbf{x}$) is a Sturmian set.

Definition

A *Sturmian* set $S \subset \mathcal{A}^*$ is a factorial set such that $p_n = \mathrm{Card}\,(S \cap \mathcal{A}^n) = n + 1$.

n :	0
p_n :	1



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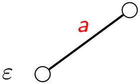

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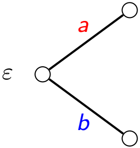
p_n :	1	2



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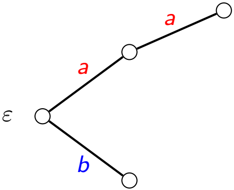

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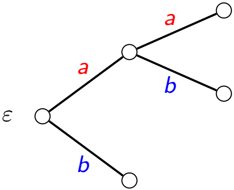

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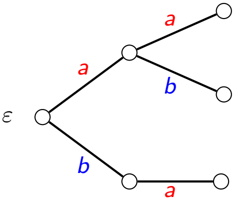
n :	0	1	2
p_n :	1	2	3



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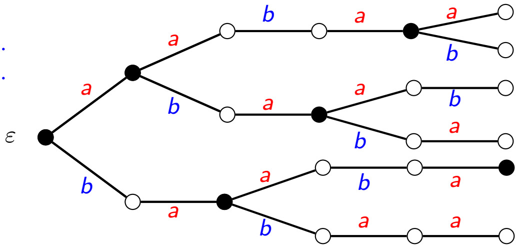
n :	0	1	2	3	4	5	$\cdots$
p_n :	1	2	3	4	5	6	$\cdots$



2-coded Fibonacci

$$\mathbf{x} = abaababaabaababa \cdots$$

$$\gamma_2 : \begin{cases} aa & \mapsto & u \\ ab & \mapsto & v \\ ba & \mapsto & w \end{cases}$$

2-coded Fibonacci

$$\mathsf{x} = \boxed{ab}\,aababaabaababa\;\cdots$$

$$\gamma_2(\mathsf{x}) = v$$

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Arnoux-Rauzy sets

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An *Arnoux-Rauzy* set is a factorial set closed by reversal with $p_n = (\mathrm{Card}\,(\mathcal{A}) - 1)n + 1$ having a unique right special factor for each length.

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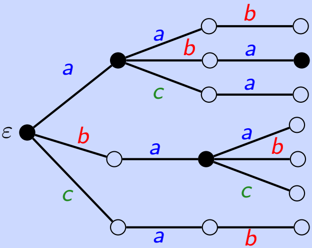
Example (Tribonacci)

Factors of the fixed point $\psi^\omega(a)$ of the morphism $\psi : a \mapsto ab, \quad b \mapsto ac, \quad c \mapsto a.$



n :	0	1	2	3	⋯
p_n :	1	3	5	7	⋯

$$p_n = 2n + 1$$

2-coded Fibonacci

$$\gamma_2(\mathbf{x}) = v\ u\ w\ w\ v\ u\ w\ w\ \cdots$$

Is the set of factors of $\gamma_2(\mathbf{x})$ an Arnoux-Rauzy set?

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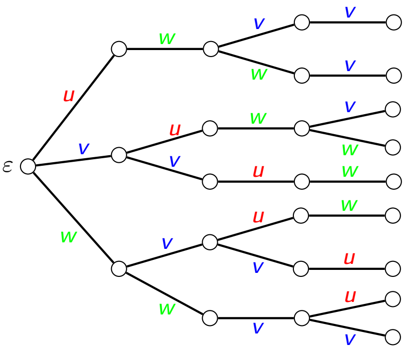

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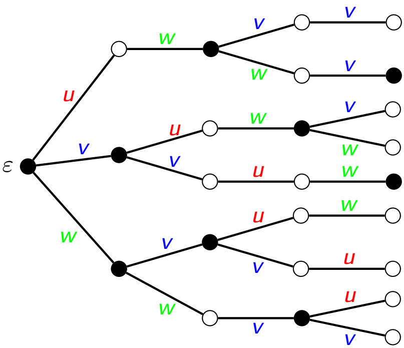

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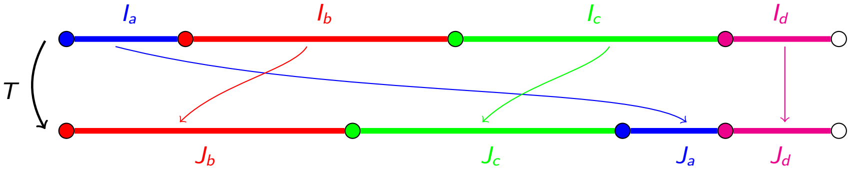
Interval exchanges

Let $(I_\alpha)_{\alpha\in\mathcal{A}}$ and $(J_\alpha)_{\alpha\in\mathcal{A}}$ be two partitions of $[0,1[$.
An *interval exchange transformation* (IET) is a map $T : [0,1[\to [0,1[$ defined by

$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha.$$



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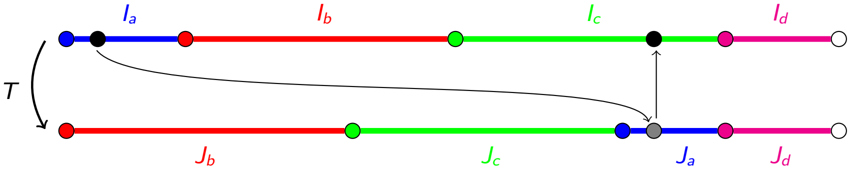

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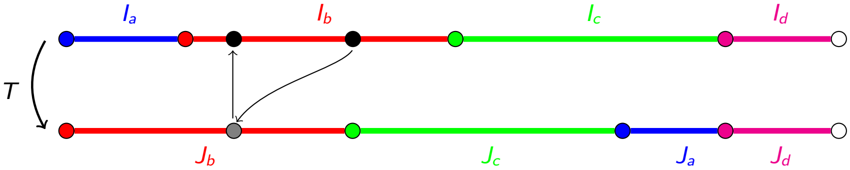

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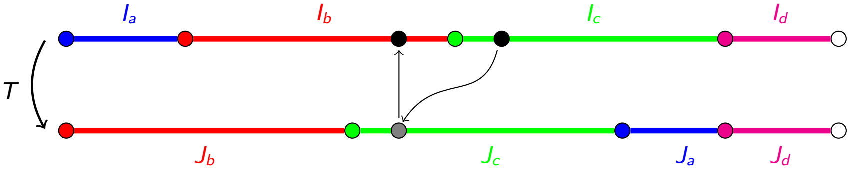

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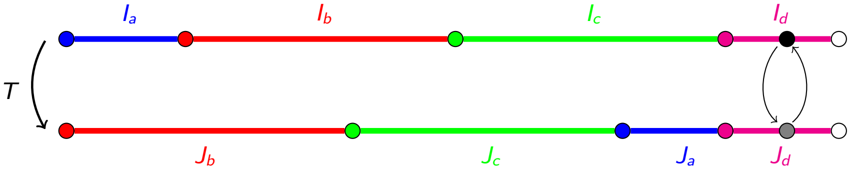

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T is said to be *minimal* if for any point $z \in [0,1[$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[0,1[$.

T is said *regular* if the orbits of the non-zero separation points are infinite and disjoint.

Theorem [Keane (1975)]

A regular interval exchange transformation is minimal.

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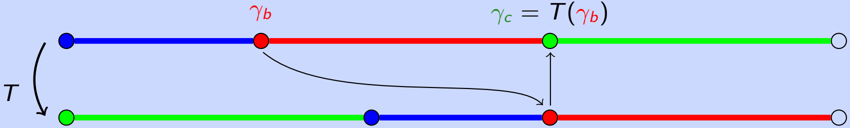
Theorem [Keane (1975)]

A regular interval exchange transformation is minimal.

Example (the converse is not true)

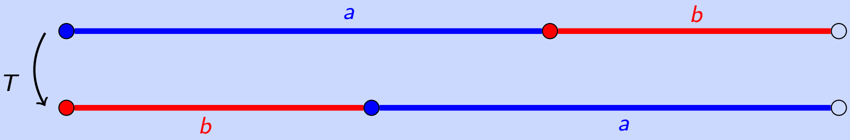


Interval exchanges

The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in \mathcal{A}^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha .$$

Example (Fibonacci, $z = (3-\sqrt{5})/2$)



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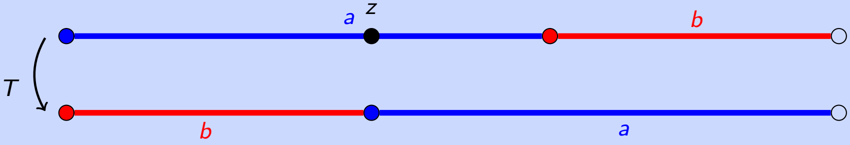

$$\Sigma_T(z) = a$$

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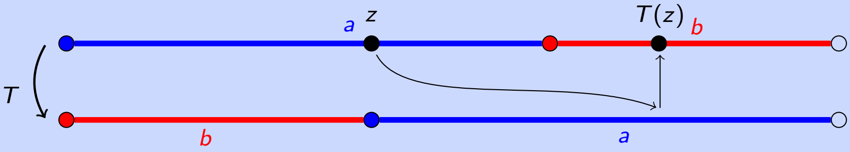

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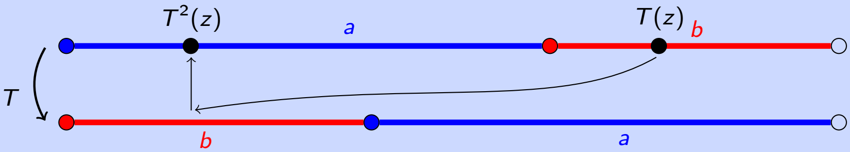

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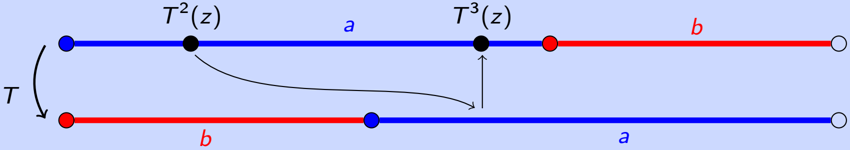

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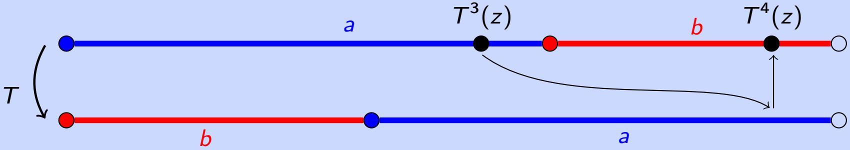

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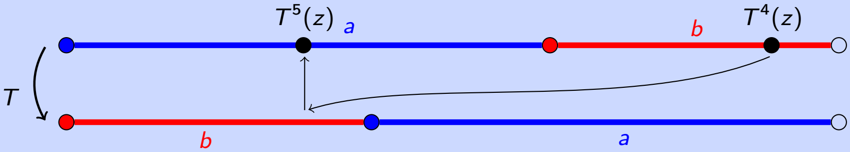
Example (Fibonacci, $z = (3-\sqrt{5})/2$)



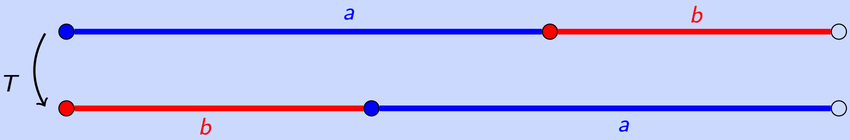
$$\Sigma_T(z) = abaaba\cdots$$

Interval exchanges

The set $\mathcal{L}(T) = \bigcup_{z \in [0,1[} \mathrm{Fac}\,(\Sigma_T(z))$ is said a (*minimal*, *regular*) *interval exchange set*.

Remark. If T is minimal, $\mathrm{Fac}\,(\Sigma_T(z))$ does not depend on the point z.

Example (Fibonacci)



$$\mathcal{L}(T) = \left\{\varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab\ \dots\right\}$$

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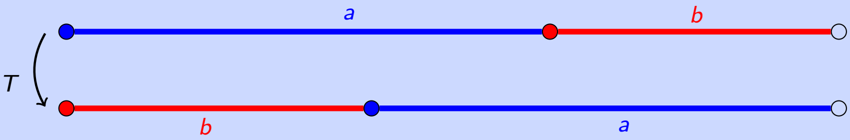
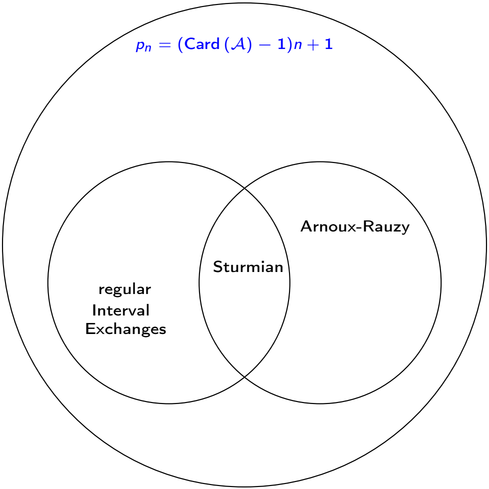


$$\mathcal{L}(T) = \left\{\varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab\ \ldots\right\}$$

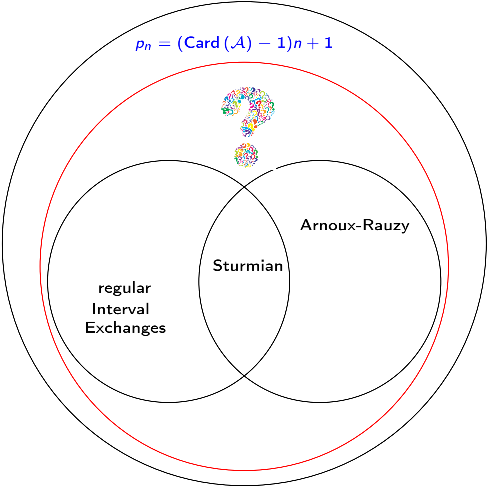
Proposition

Regular interval exchange sets have factor complexity $p_n = (\mathrm{Card}\,(\mathcal{A}) - 1)n + 1$.

Arnoux-Rauzy and Interval exchanges



Arnoux-Rauzy and Interval exchanges



Extension graphs

The *extension graph* of a word $w \in \mathcal{L}$ (w.r.t. $\mathcal{L}$) is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

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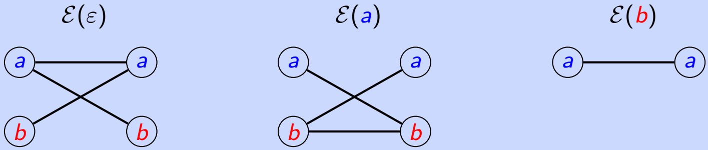

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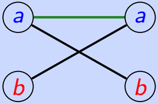
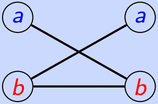
$\mathcal{E}(\varepsilon)$



$\mathcal{E}(a)$



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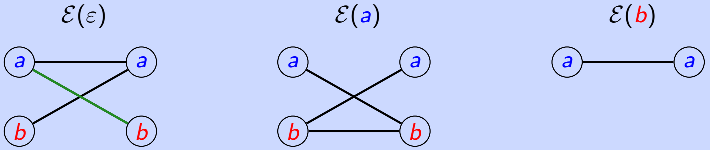

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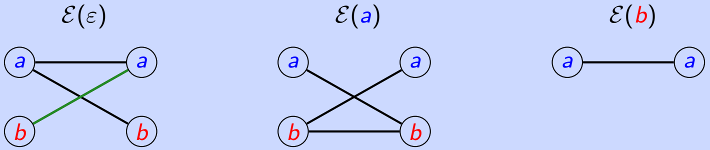

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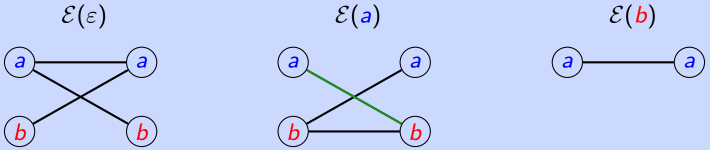

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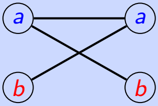

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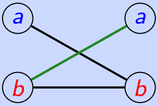
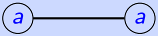


$\mathcal{E}(b)$



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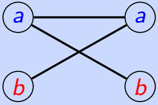

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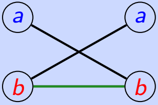

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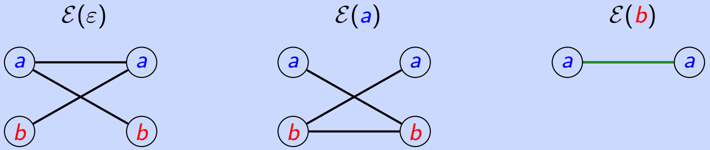

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The *multiplicity* of a word w is the quantity

$$m(w) = \mathrm{Card}\,(B(w)) - \mathrm{Card}\,(L(w)) - \mathrm{Card}\,(R(w)) + 1.$$

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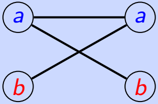
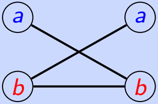
$\mathcal{E}(\varepsilon)$



$\mathcal{E}(a)$



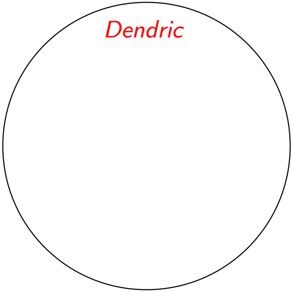
$\mathcal{E}(b)$

$m(a) = 3 - 2 - 2 + 1 = 0$

Dendric and neutral sets

Definition

A language $\mathcal{L}$ is called (purely) *dendric* if the graph $\mathcal{E}(w)$ is a tree for any $w \in \mathcal{L}$.



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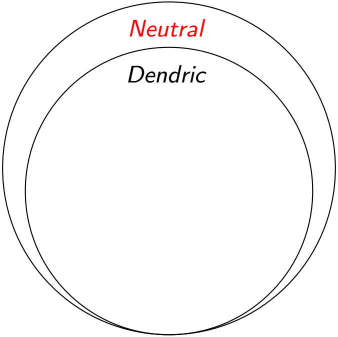

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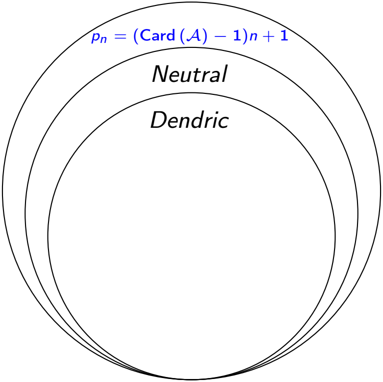

[*using* Cassaigne: "Complexité et facteurs spéciaux" (1997).]

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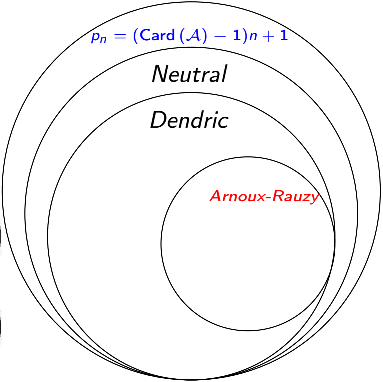

[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone: "Acyclic, connected and tree sets" (2014).]

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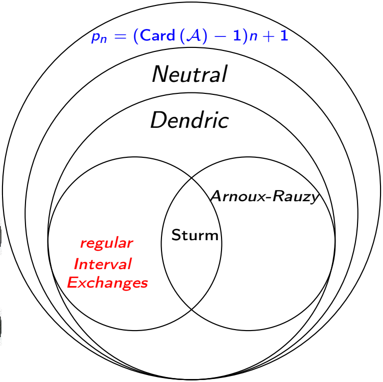

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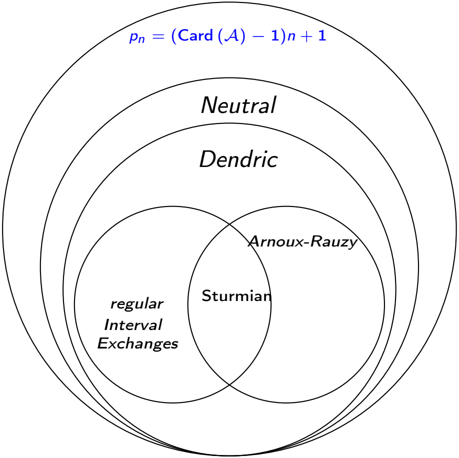

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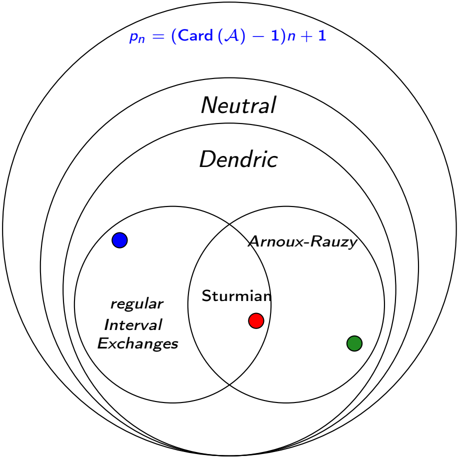
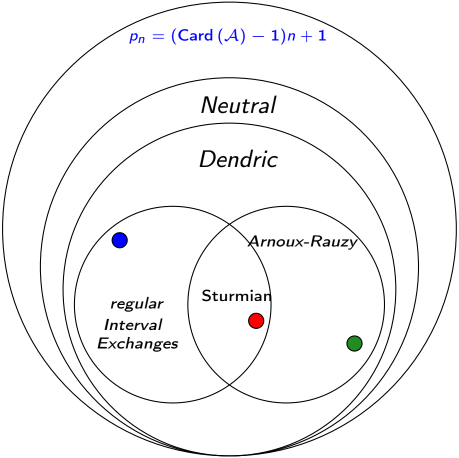


- Fibonacci
- Tribonacci
- regular IE

Dendric and neutral sets



- • Fibonacci
- ? 2-coded Fibonacci
- • Tribonacci
- ? 2-coded Tribonacci
- • regular IE
- ? 2-coded regular IE

2-coded Fibonacci

$$\mathbf{x} = abaababaabaababa \cdots$$

$$\gamma_2(\mathbf{x}) = vwuvwvwvwuv \cdots$$

$$\gamma_2 : \begin{cases} aa & \mapsto & u \\ ab & \mapsto & v \\ ba & \mapsto & w \end{cases}$$

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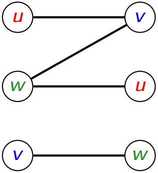
$\mathcal{E}(\varepsilon)$ ✗



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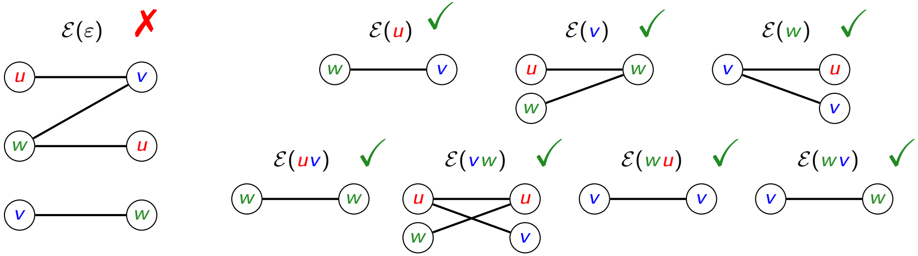

Eventually dendric sets

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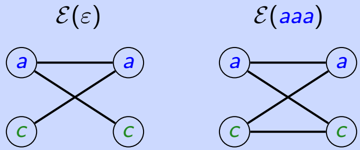
Example (coding of Tribonacci)

Let us consider the set $\alpha(S)$, where $\alpha : a, b \mapsto a, \quad c \mapsto c.$



The extension graph of all words of length at least 4 is a tree. (Just trust me!)

Shift spaces

$\mathcal{A}^{\mathbb{Z}} = \{(x_n) \mid n \in \mathbb{Z}\}$ with the natural product topology.

The *shift transformation* is the function

$$\begin{array}{rccc} \sigma : & \mathcal{A}^{\mathbb{Z}} & \to & \mathcal{A}^{\mathbb{Z}} \\ & (x_n)_{n\in\mathbb{Z}} & \mapsto & (x_{n+1})_{n\in\mathbb{Z}} \end{array}$$

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$$\mathbf{x} = \cdots ab.abaababaabaababaababaabaab\cdots$$

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The pair (X, σ), with X a closed σ-invariant subset of $\mathcal{A}^{\mathbb{Z}}$ is called a *shift space*.

Example (Fibonacci, but on two sides)

The *Fibonacci shift space* is the set $X = \overline{\mathcal{O}(\mathbf{x})} = \overline{\{\sigma^n(\mathbf{x}) \mid n \in \mathbb{Z}\}} \subset \mathcal{A}^{\mathbb{Z}}$, with

$$\mathbf{x} = \cdots ab.abaababaabaababaababaabaab\cdots$$

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The *Fibonacci shift space* is the set $X = \overline{\mathcal{O}(\mathbf{x})} = \overline{\{\sigma^n(\mathbf{x}) \mid n \in \mathbb{Z}\}} \subset \mathcal{A}^{\mathbb{Z}}$, with

$$\mathbf{x} = \cdots ab.abaababaabaababaababaabaab\cdots$$

The *language* of a shift space (X, σ) is the set $\mathcal{L}(X) = \bigcup_{\mathbf{x}\in X} \mathrm{Fac}(\mathbf{x})$.

Shift spaces

$\mathcal{A}^{\mathbb{Z}} = \{(x_n) \mid n \in \mathbb{Z}\}$ with the natural product topology.

The *shift transformation* is the function

$$\begin{array}{rccc} \sigma: & \mathcal{A}^{\mathbb{Z}} & \to & \mathcal{A}^{\mathbb{Z}} \\ & (x_n)_{n\in\mathbb{Z}} & \mapsto & (x_{n+1})_{n\in\mathbb{Z}} \end{array}$$

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A shift space (X, σ) is *(eventually) dendric shift* if its language is (eventually) dendric.

Eventually dendric shifts

Complexity

Let us consider the function $s_n = p_{n+1} - p_n$.

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Proposition [D., Perrin (2019)]

Let X be eventually dendric. Then s_n is eventually constant on $\mathcal{L}(X)$.

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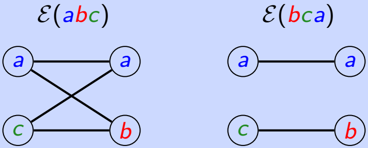
Example (the converse is not true)

The *Chacon ternary shift* is the shift arising from the morphism $\varphi : \begin{cases} a \mapsto aabc \\ b \mapsto bc \\ c \mapsto abc \end{cases}$.

One has $p_n = 2n + 1$ ($\Rightarrow s_n = 2$). BUT for infinitly many pairs of words:



Asymptotic equivalence

$\mathbf{x}^+ \in \mathcal{A}^{\mathbb{N}}$ is *tail* of $\mathbf{x} \in \mathbb{Z}$ if $\mathbf{x} = \mathbf{x}^-\mathbf{x}^+$ for some $\mathbf{x}^- \in \mathcal{A}^{-\mathbb{N}}$.

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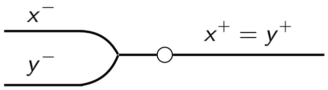
$\mathbf{x}, \mathbf{y} \in \mathcal{A}^{\mathbb{Z}}$ are *right asymptotic equivalent* if they have a common tail.



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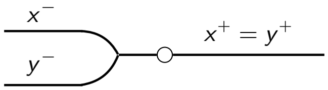
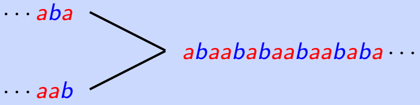


A *right asymptotic class* is an equivalence class C having more than one orbit.

Example (Fibonacci)



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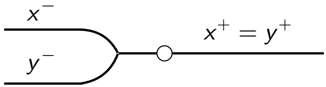
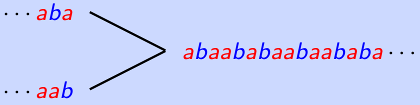


A *right asymptotic class* is an equivalence class C having more than one orbit.
For a shift X, we denote $\omega(X) = \sum_C \text{Card}\,(o(C)) - 1$.

Example (Fibonacci)



$$\omega(X) = 1$$

Asymptotic equivalence

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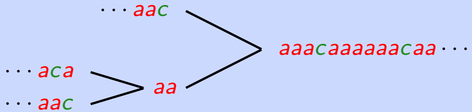
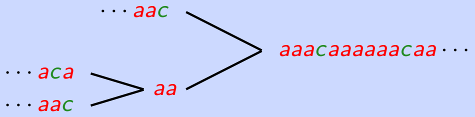
Example (coding of Tribonacci) $\omega(X) = 2$



Asymptotic equivalence

$$\omega(X) = \sum_{C} \mathrm{Card}\,(o(C)) - 1.$$

Example (coding of Tribonacci) $\omega(X) = 2 = s_n$



Theorem [D., Perrin (2019)]

A shift space X is eventually dendric if and only if:

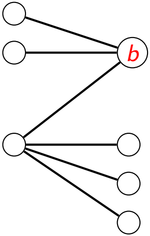
1. s_n is eventually constant on $\mathcal{L}(X)$, and
2. $\lim s_n = \omega(X)$.

Eventually dendric shifts

Theorem [D., Perrin (2019)]

A shift X is eventually dendric if and only if there exists $N \geq 0$ s.t. any left-special word $w \in \mathcal{L}(X)$ of length at least N has exactly ONE right extension wb that is left-special.

$\mathcal{E}(w)$



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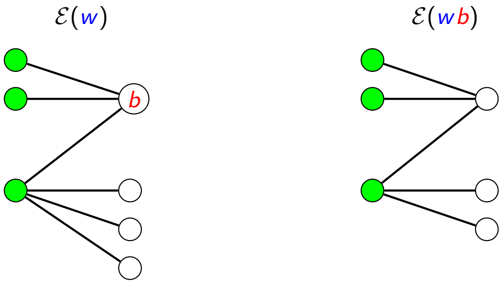

Conjugacy

Let $X \subset \mathcal{A}^{\mathbb{Z}}$ and $Y \subset \mathcal{B}^{\mathbb{Z}}$.

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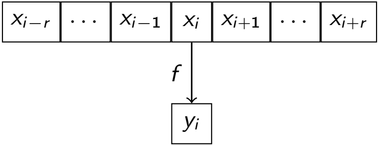
$$\phi((\mathbf{x})_i) = f(x_{i-r} \cdots x_{i+r}).$$



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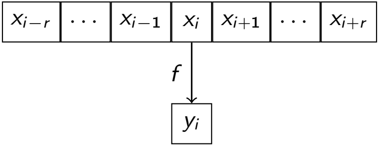

X, Y are said to be *conjugate* when there is a bijective sliding block code $\phi : X \to Y$.

Example of conjugacies

▷ *k-th higher block codes*,

$$\gamma_k : \quad X \longrightarrow X^{(k)} \subset \mathcal{A}_k^{\mathbb{Z}}$$
$$(x_n)_n \mapsto (y_n)_n \qquad\qquad y_n = f(x_n \cdots x_{n+k-1})$$

Example (Fibonacci)

$$\gamma_k : \quad X \longrightarrow X^{(2)}$$

$$(x_n) \mapsto (y_n) \qquad f : \begin{cases} \underline{a}a \mapsto u \\ \underline{a}b \mapsto v \\ \underline{b}a \mapsto w \end{cases}$$

Example of conjugacies

- *k-th higher block codes*,

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- *alphabetic morphisms*,

$$\alpha : \mathcal{A}^* \to \mathcal{B}^* \quad \text{s.t.} \quad \alpha(\mathcal{A}) \subset \mathcal{B}.$$

Example (Tribonacci, $\mathcal{A} = \{a, b, c\}$, $\mathcal{B} = \{a, c\}$)

$$\alpha : \quad X \longrightarrow Y$$
$$(x_n) \mapsto (y_n) \qquad \begin{cases} a \mapsto a \\ b \mapsto a \\ a \mapsto c \end{cases}$$

Conjugacy

Theorem [D., Perrin (2019)]

The family of eventually dendric sets is closed under conjugacy.

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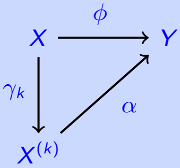
Proof.



Open questions

- Closure under taking factors ?

  [Y is a *factor* of X, if there is a sliding block (not necessary bijective) $\phi : X \to Y$]

- Subgroup generated by sets of return words in an eventually dendric set ?

  [For a dendric set, $\mathcal{R}(w)$ is a basis of the free group on $\mathcal{A}$.]

- Decidability of the (eventually) dendric condition.

  [Work in progress with Revekka Kyriakoglou and Julien Leroy]

Спасибо!