Eventually dendric shifts

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CZECH TECHNICAL UNIVERSITY IN PRAGUE

Journées SDA2 2019

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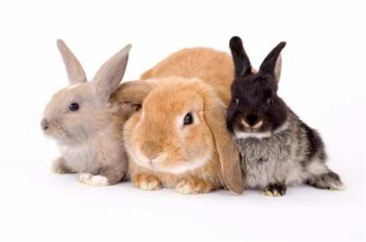
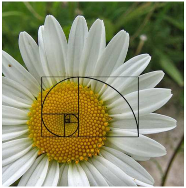
Fibonacci

$$\mathbf{x} = \cdots ab.abaababaabaababa\cdots$$

$$\mathbf{x} = \lim_{n\to\infty} \varphi^n(a) \qquad \text{where} \qquad \varphi : \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$





Fibonacci

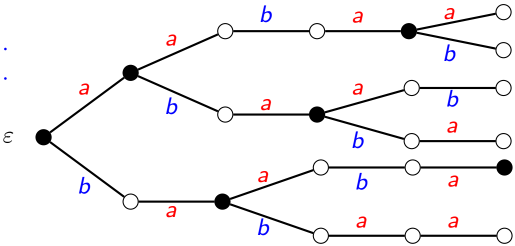
$$\mathbf{x} = \cdots ab.abaababaabaababa \cdots$$

The *Fibonacci set* (set of factors of $\mathbf{x}$) is a Sturmian set.

Definition

A *Sturmian* set $S \subset \mathcal{A}^*$ is a factorial set such that $p_n = \mathrm{Card}\,(S \cap \mathcal{A}^n) = n + 1$.

n :	0	1	2	3	4	5	$\cdots$
p_n :	1	2	3	4	5	6	$\cdots$



Shift spaces

The *shift transformation* is the function

$$\begin{array}{rccc} \sigma : & \mathcal{A}^{\mathbb{Z}} & \to & \mathcal{A}^{\mathbb{Z}} \\ & (x_n)_{n\in\mathbb{Z}} & \mapsto & (x_{n+1})_{n\in\mathbb{Z}} \end{array}$$

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The pair (X, σ), with X a closed σ-invariant subset of $\mathcal{A}^{\mathbb{Z}}$ is called a *shift space*.

Example (Fibonacci, but on two sides)

The *Fibonacci shift space* is the set $X = \overline{\mathcal{O}(\mathbf{x})} = \overline{\{\sigma^n(\mathbf{x}) \mid n \in \mathbb{Z}\}} \subset \mathcal{A}^{\mathbb{Z}}$, with

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The *language* of a shift space (X, σ) is the set $\mathcal{L}(X) = \bigcup_{\mathbf{x}\in X} \mathrm{Fac}(\mathbf{x})$.

2-coded Fibonacci

$$\mathbf{x} = \cdots\ ab\ .\ ab\ aa\ ba\ ba\ ab\ aa\ ba\ ba\ \cdots$$

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Arnoux-Rauzy sets

Definition

An *Arnoux-Rauzy* set is a factorial set closed by reversal with $p_n = (\mathrm{Card}\,(\mathcal{A}) - 1)n + 1$ having a unique right special factor for each length.

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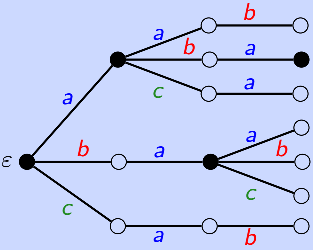
Example (Tribonacci)

Factors of the fixed point $\psi^\omega(a)$ of the morphism $\psi : a \mapsto ab, \quad b \mapsto ac, \quad c \mapsto a.$



n :	0	1	2	3	$\cdots$
p_n :	1	3	5	7	$\cdots$

$p_n = 2n + 1$

2-coded Fibonacci

$$f^{-1}(\mathbf{x}) = \cdots\ v\ .\ v\ u\ w\ w\ v\ u\ w\ w\ \cdots$$

Is the set of factors of $f^{-1}(\mathbf{x})$ an Arnoux-Rauzy set ?

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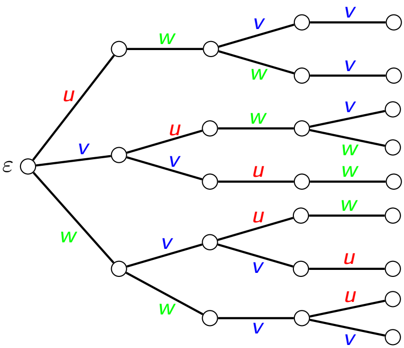

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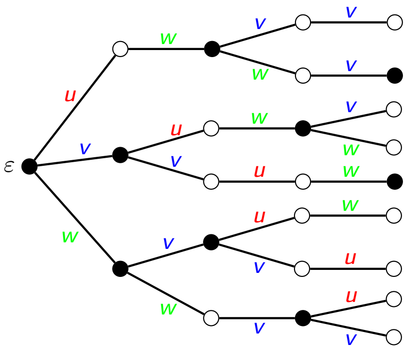
Is the set of factors of $f^{-1}(\mathbf{x})$ an Arnoux-Rauzy set ? No !



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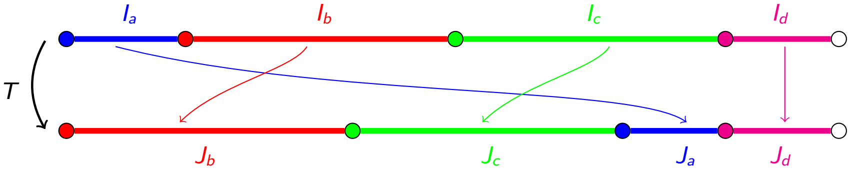
Interval exchanges

Let $(I_\alpha)_{\alpha\in\mathcal{A}}$ and $(J_\alpha)_{\alpha\in\mathcal{A}}$ be two partitions of $[0,1[$.
An *interval exchange transformation* (IET) is a map $T:[0,1[\to[0,1[$ defined by

$$T(z)=z+y_\alpha \quad \text{if } z\in I_\alpha.$$



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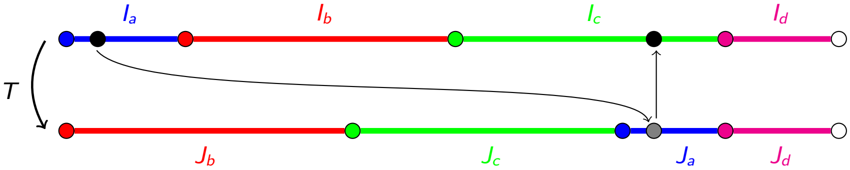

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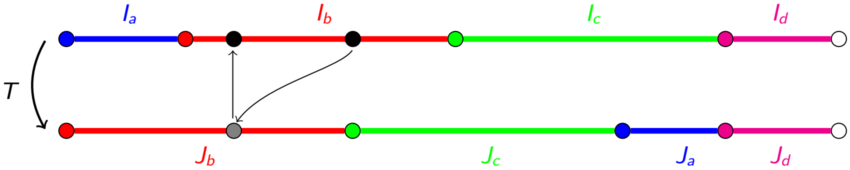

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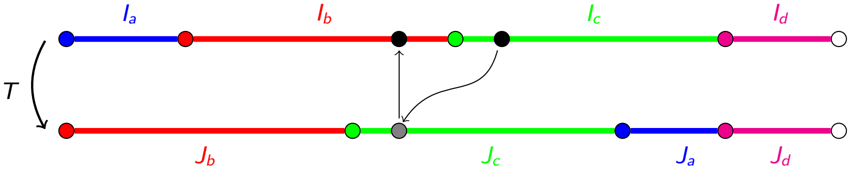

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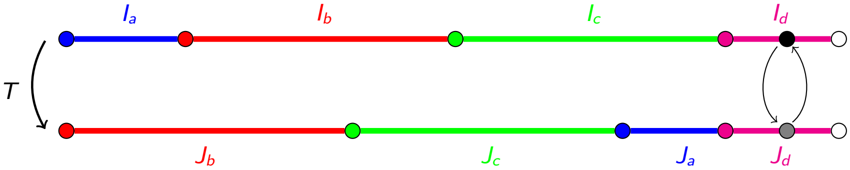

Interval exchanges

T is said to be *minimal* if for any point $z \in [0,1[$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[0,1[$.

T is said *regular* if the orbits of the non-zero separation points are infinite and disjoint.

Theorem [Keane (1975)]

A regular interval exchange transformation is minimal.

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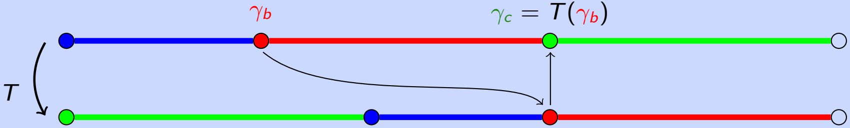
Example (the converse is not true)

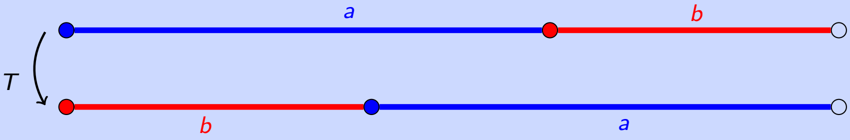


Interval exchanges

The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = \cdots a_{-1}.a_0 a_1 \cdots \in \mathcal{A}^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha.$$

Example (Fibonacci, $z = (3-\sqrt{5})/2$)



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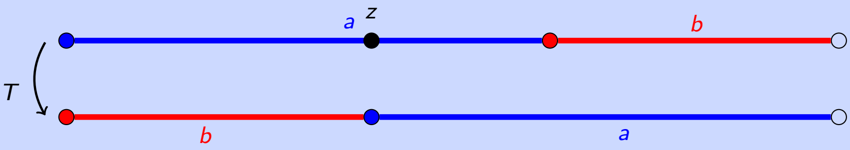

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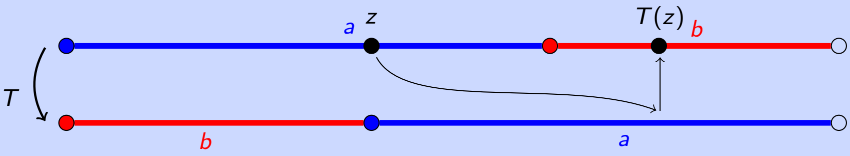

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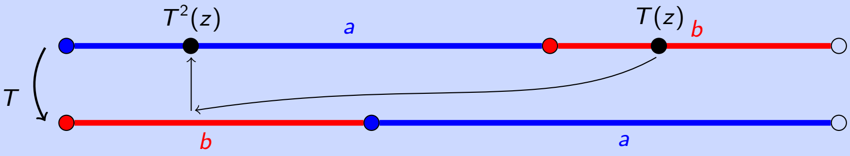

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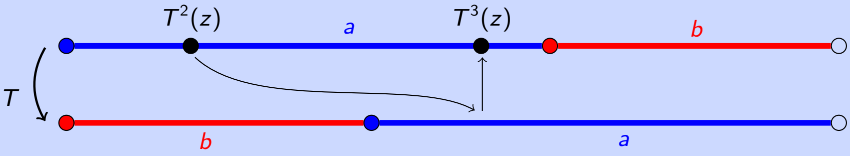

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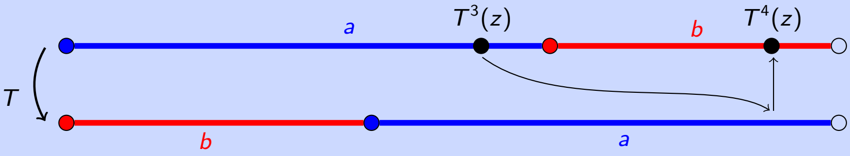

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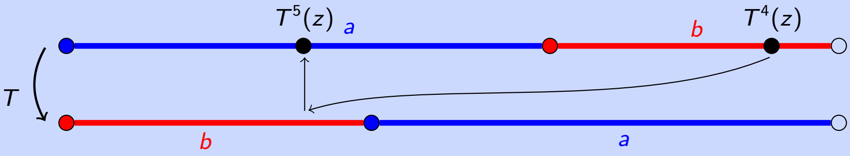

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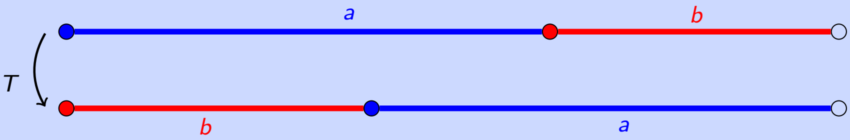
The set $\mathcal{L}(T) = \bigcup_{z \in [0,1[} \mathrm{Fac}(\Sigma_T(z))$ is said a (*minimal*, *regular*) *interval exchange set*.

Remark. If T is minimal, $\mathrm{Fac}(\Sigma_T(z))$ does not depend on the point z.

Example (Fibonacci)



$$\mathcal{L}(T) = \{\varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab\ \dots\}$$

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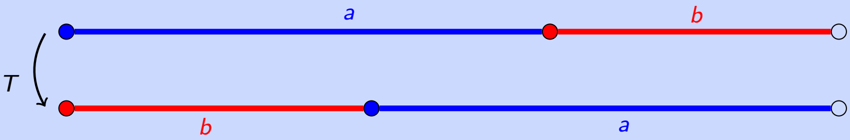

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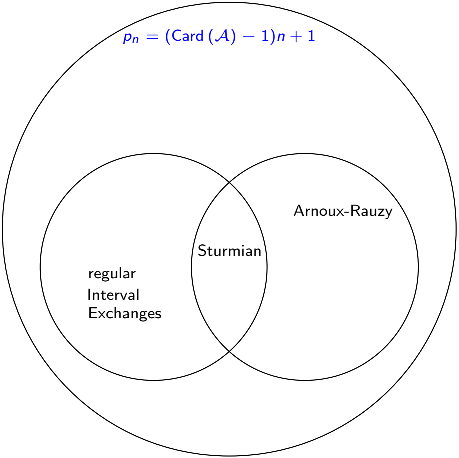
Proposition

Regular interval exchange sets have factor complexity $p_n = (\mathrm{Card}\,(\mathcal{A}) - 1)n + 1$.

Arnoux-Rauzy and Interval exchanges



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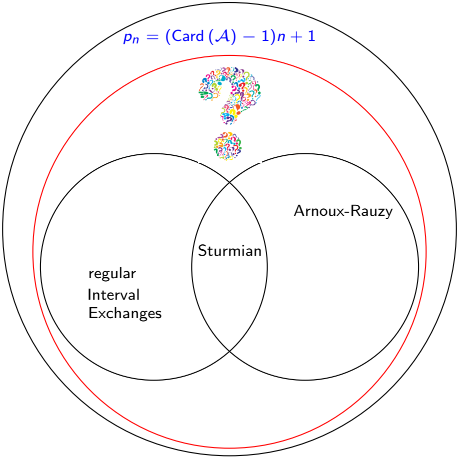

Extension graphs

The *extension graph* of a word $w \in \mathcal{L}(X)$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{a \in \mathcal{A} \mid aw \in \mathcal{L}(X)\} \\ R(w) &= \{a \in \mathcal{A} \mid wa \in \mathcal{L}(X)\} \\ B(w) &= \{(a,b) \in \mathcal{A} \times \mathcal{A} \mid awb \in \mathcal{L}(X)\} \end{aligned}$$

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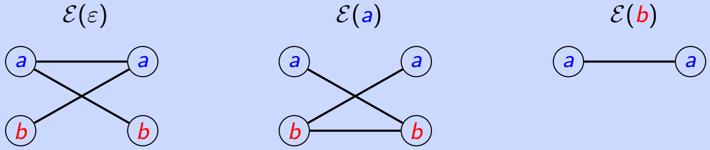

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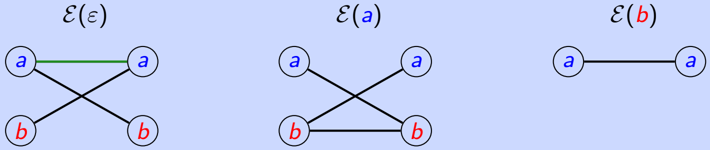

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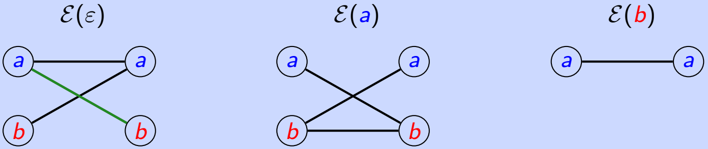

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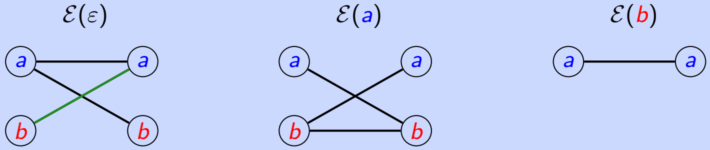

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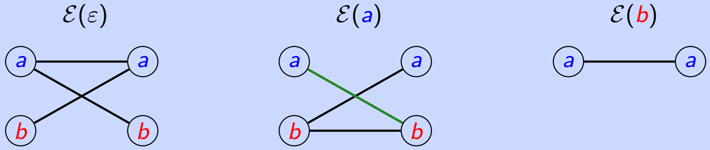
Example (Fibonacci, $\mathcal{L}(X) = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, bab, \ldots\}$)



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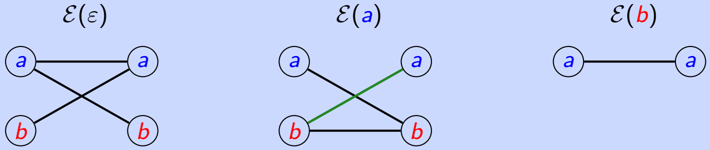

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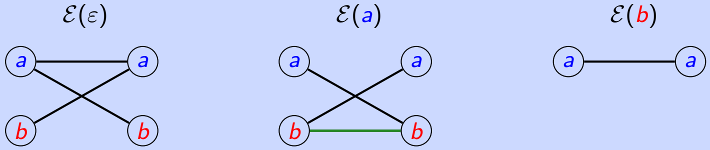

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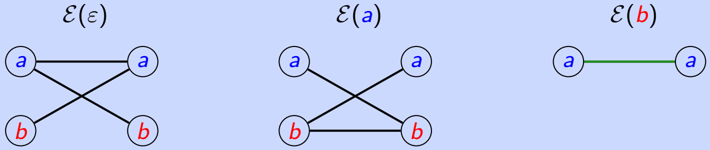

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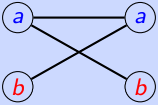
The *multiplicity* of a word w is the quantity

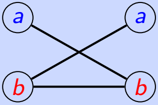
$$m(w) = \text{Card}\,(B(w)) - \text{Card}\,(L(w)) - \text{Card}\,(R(w)) + 1.$$

Example (Fibonacci, $\mathcal{L}(X) = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, bab, \ldots\}$)

$\mathcal{E}(\varepsilon)$



$\mathcal{E}(a)$



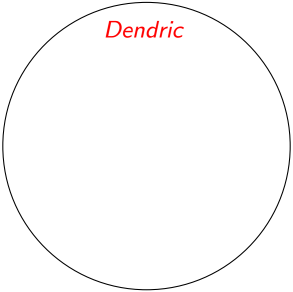
$\mathcal{E}(b)$

$m(a) = 3 - 2 - 2 + 1 = 0$

Dendric and neutral shifts

Definition

A shift space X is called a *dendric shift* if the graph $\mathcal{E}(w)$ is a tree for any $w \in \mathcal{L}(X)$.



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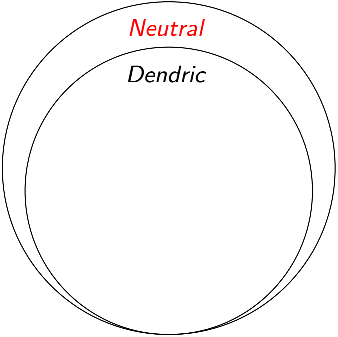

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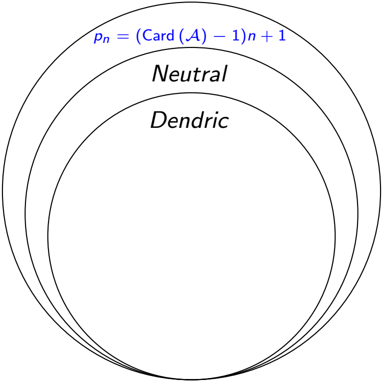

[*using* Cassaigne : "**Complexité et facteurs spéciaux**" (1997).]

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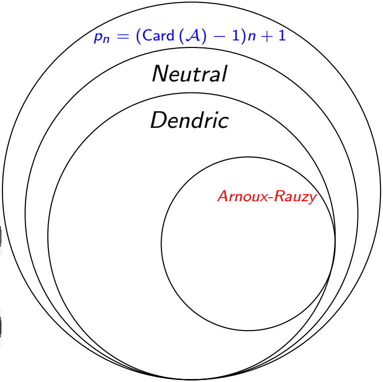

[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone : "**Acyclic, connected and tree sets**" (2014).]

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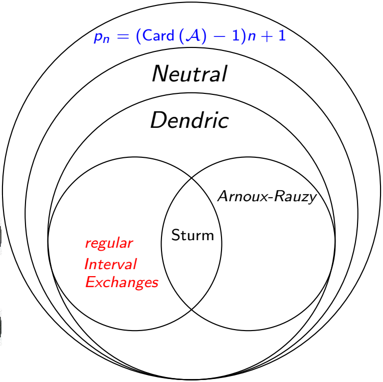

[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone : "**Bifix codes and interval exchanges**" (2015).]

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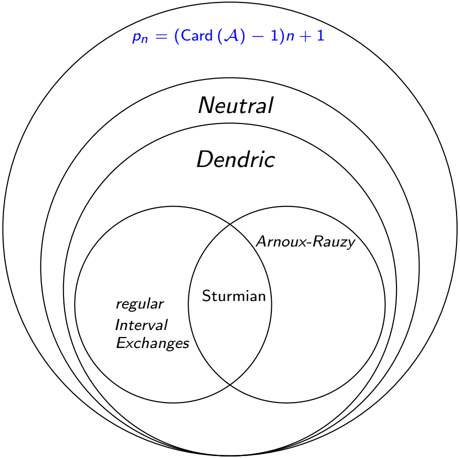

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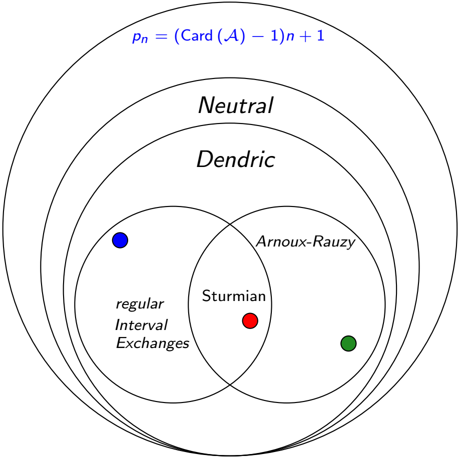
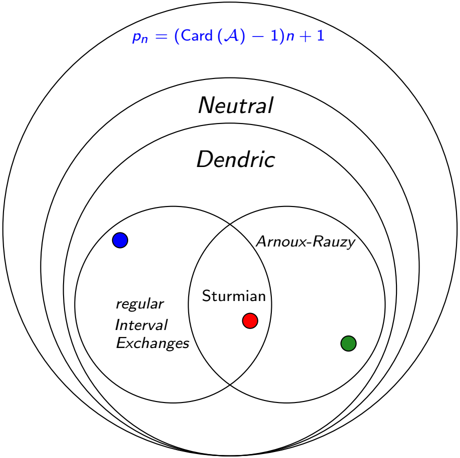


- Fibonacci
- Tribonacci
- regular IE

Dendric and neutral shifts



- • Fibonacci
- ? 2-coded Fibonacci
- • Tribonacci
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- • regular IE
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Bifix codes

Definition

A *bifix code* is a set $B \subset \mathcal{A}^+$ of nonempty words that does not contain any proper prefix or suffix of its elements.

Example

- ✓ $\{aa, ab, ba\}$
- ✓ $\{aa, ab, bba, bbb\}$
- ✓ $\{ac, bcc, bcbca\}$
- ✗ { choc, chocolate, vanille }
- ✗ { arbre, feuille, marbre }
- ✗ { aise, fraise, frai }

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A bifix code $B \subset S$ is *S-maximal* if it is not properly contained in a bifix code $C \subset S$.

Example (Fibonacci)

The set $B = \{aa, ab, ba\}$ is an S-maximal bifix code.
It is not an $\mathcal{A}^*$-maximal bifix code, since $B \subset B \cup \{bb\}$.

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A *coding morphism* for a bifix code $B \subset A^+$ is a morphism $f : \mathcal{B}^* \to \mathcal{A}^*$ which maps bijectively $\mathcal{B}$ onto B.

Example

The map $f : \{u, v, w\}^* \to \{a, b\}^*$ is a coding morphism for $B = \{aa, ab, ba\}$.

$$f : \begin{cases} u \mapsto aa \\ v \mapsto ab \\ w \mapsto ba \end{cases}$$

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When S is factorial and B is an S-maximal bifix code, the set $f^{-1}(S)$ is called a *maximal bifix decoding* of S.

Irreducibility and minimality

Definition

A shift X is *irreducible* if $\mathcal{L}(X)$ is *recurrent*, that is if for every $u,v \in \mathcal{L}(X)$ there is a $w \in \mathcal{L}(X)$ such that uwv is in $\mathcal{L}(X)$.

Example (Fibonacci)

$$x = \cdots ab.abaababaabaababaababaabaababa\cdots$$

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X is *minimal* if $\mathcal{L}(X)$ is *uniformly recurrent*, that is if for every $u \in S$ there exists an $n \in \mathbb{N}$ such that u is a factor of every word of length n in S.

Example (Fibonacci)

$$x = \cdots ab.\underbrace{abaa}_{4}ba\underbrace{baab}_{4}\underbrace{aaba}_{4}baababaaba\underbrace{abab}_{4}a\cdots$$

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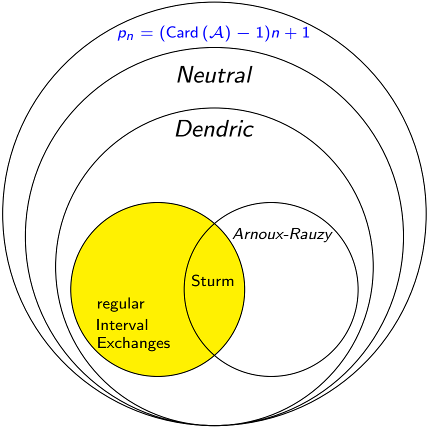
Proposition

Minimal $\implies$ irreducible.

Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014)]

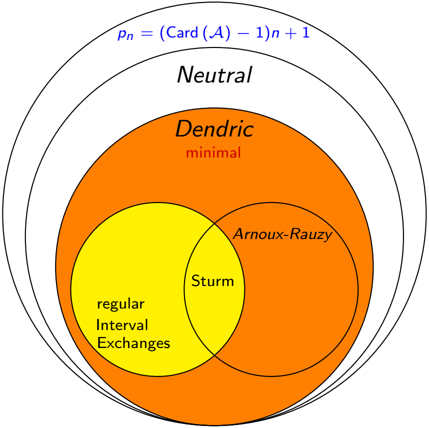
The family of regular interval exchanges shifts is closed under maximal bifix decoding.



Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015)]

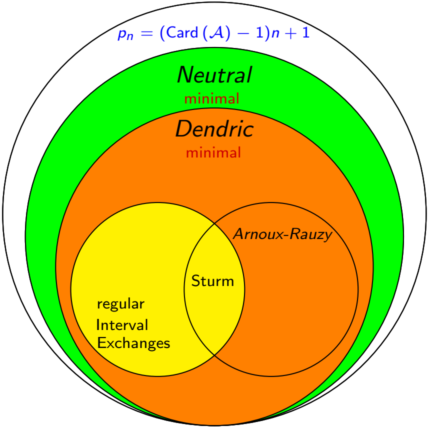
The family of *minimal* dendric shifts is closed under maximal bifix decoding.



Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015) ; D., Perrin (2016)]

The family of *minimal* neutral shifts is closed under maximal bifix decoding.



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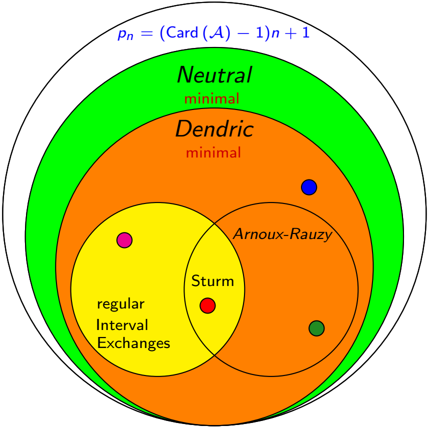

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A question by Fabien Durand

$$\mathbf{x} = \cdots \ ab.abaababaabaababa \ \cdots$$

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$$\begin{cases} u & \mapsfrom & aa \\ v & \mapsfrom & ab \\ w & \mapsfrom & ba \end{cases}$$

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$$\mathbf{x} = \cdots \boxed{ab}.abaababaabaababa \cdots$$

$$\sigma(\mathbf{x}) = v$$

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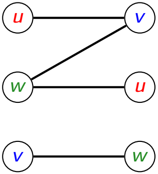
$\mathcal{E}(\varepsilon)$ ✗



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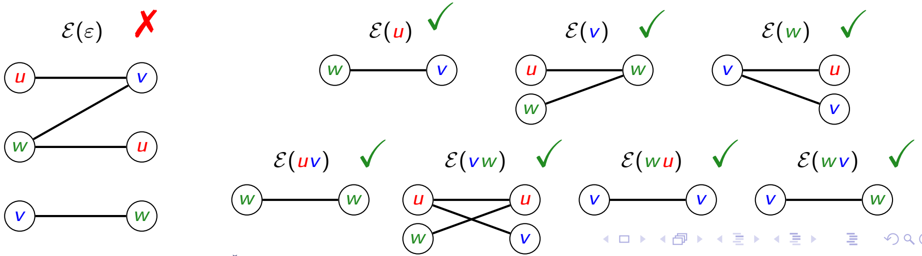

Eventually dendric shifts

Definition

A shift space X is called a *eventually dendric* with threshold $m \geq 0$ if the graph $\mathcal{E}(w)$ is a tree for any $w \in \mathcal{L}(X)$ s.t. $|w| \geq m$.

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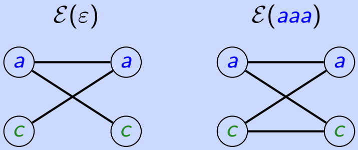
Example (coding of Tribonacci)

Let us consider the set $\alpha(S)$, where $\alpha : a, b \mapsto a, \quad c \mapsto c$.



The extension graph of all words of length at least 4 is a tree. (Just trust me !)

Eventually dendric shifts

Complexity

Let us consider the function $s_n = p_{n+1} - p_n$.

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Proposition [D., Perrin (2019)]

Let X be eventually dendric. Then s_n is eventually constant on $\mathcal{L}(X)$.

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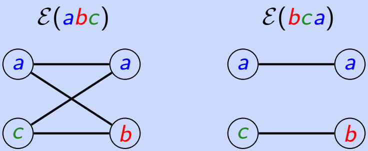
Example (the converse is not true)

The *Chacon ternary shift* is the shift arising from the morphism $\varphi : \begin{cases} a \mapsto aabc \\ b \mapsto bc \\ c \mapsto abc \end{cases}$.

One has $p_n = 2n + 1$ ($\Rightarrow s_n = 2$). But for infinitly many pairs of words :

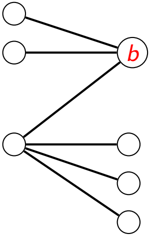


Eventually dendric shifts

Theorem [D., Perrin (2019)]

A shift X is eventually dendric if and only if there exists $N \geq 0$ s.t. any left-special word $w \in \mathcal{L}(X)$ of length at least N has exactly ONE right extension wb that is left-special.

$\mathcal{E}(w)$



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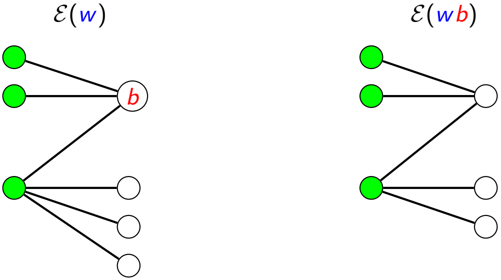
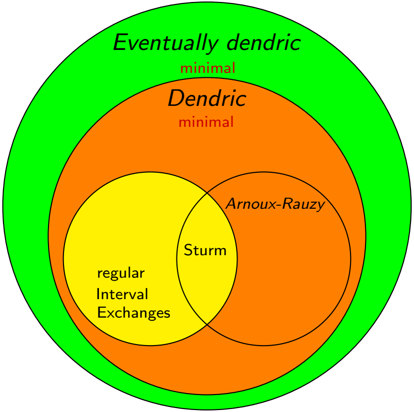


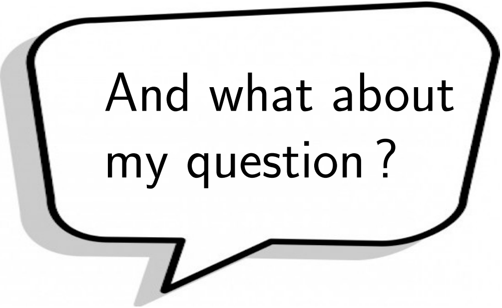
Eventually dendric shifts

Maximal bifix decoding

Theorem [D., Perrin (2019)]

The family of minimal eventually dendric sihfts (with threshold m) is closed under maximal bifix decoding.




And what about my question ?

Conjugacy

A map $\phi : X \to Y$ is called a *sliding block code* if it is continuous and $\phi \circ \sigma_X = \sigma_Y \circ \phi$.

$$\begin{array}{ccc} & \sigma_X & \\ X & \longrightarrow & X \\ \phi\downarrow & & \downarrow\phi \\ Y & \longrightarrow & Y \\ & \sigma_Y & \end{array}$$

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$$\begin{array}{ccc} & \sigma_X & \\ X & \longrightarrow & X \\ \phi\downarrow & & \downarrow\phi \\ Y & \longrightarrow & Y \\ & \sigma_Y & \end{array}$$

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- *k-th higher block codes*, i.e., $\gamma_k : \begin{array}{c} X \longrightarrow X^{(k)} \subset \mathcal{A}_k^{\mathbb{Z}} \\ (x_n)_n \mapsto (y_n)_n \end{array} \quad y_n = f(x_n \cdots x_{n+k-1})$

Example (Fibonacci)

$$\gamma_k : \begin{array}{c} X \longrightarrow X^{(2)} \\ \\ (x_n) \mapsto (y_n) \end{array} \qquad f : \begin{cases} \underline{a}a \mapsto u \\ \underline{a}b \mapsto v \\ \underline{b}a \mapsto w \end{cases}$$

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- ▷ *alphabetic morphisms*, i.e., $\alpha : \mathcal{A}^* \to \mathcal{B}^*$ s.t. $\alpha(\mathcal{A}) \subset \mathcal{B}$.

Example (Tribonacci, $\mathcal{A} = \{a, b, c\}$, $\mathcal{B} = \{a, c\}$)

$$\gamma_k : \begin{array}{ccc} X & \longrightarrow & Y \\ (x_n) & \mapsto & (y_n) \end{array} \qquad \begin{cases} a \mapsto a \\ b \mapsto a \\ a \mapsto c \end{cases}$$

Conjugacy

Theorem [D., Perrin (2019)]

The family of eventually dendric sets is closed under conjugacy.

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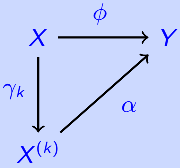
Proof.



Ergodicity of dendric shift spaces

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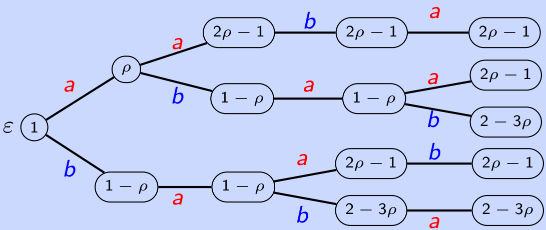
Theorem [Arnoux, Rauzy (1991)]

Shift spaces associated to Arnoux-Rauzy sets are uniquely ergodic.

Example (Fibonacci, $\rho = (\sqrt{5} - 1)/2$)

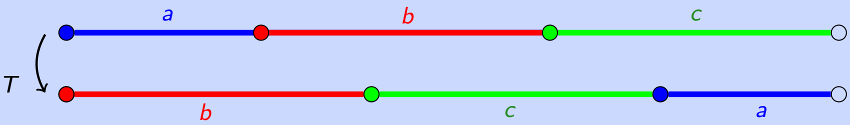


Ergodicity of dendric shift spaces

Given an interval exchange transformation T and a word $w = a_0 a_1 \cdots a_{m-1} \in \mathcal{A}^*$, let

$$I_w = I_{a_0} \cap T^{-1}(I_{a_1}) \cap \ldots \cap T^{-m+1}(I_{a_{m-1}})$$

Example



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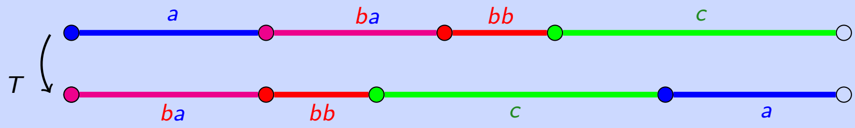

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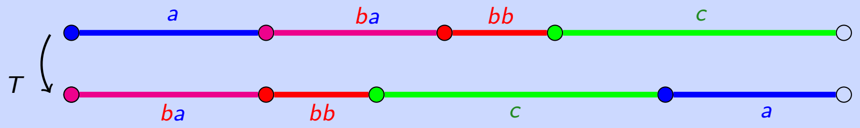

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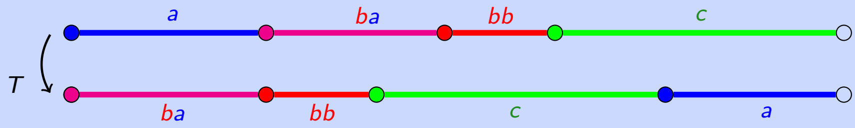

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QUESTION : Is it the only one ?

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Conjecture [Keane (1975)]

Every regular IE is uniquely ergodic.

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Corollary

Dendric shift spaces are **not** in general uniquely ergodic (even when minimal).

Ergodicity of dendric shift spaces

Theorem [Boshernitzan (1984)]

A minimal symbolic system such that $\limsup_{n\to\infty}\left(\frac{p_n}{n}\right) < 3$ is uniquely ergodic.

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Minimal dendric shift spaces over an alphabet of size ≤ 3 are uniquely ergodic.


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Minimal dendric shift spaces over an alphabet of size ≤ 3 are uniquely ergodic.

Theorem [Damron, Fickenscher (2019)]

A minimal eventually dendric shift space has at most $\frac{\sup_n(s_n)+1}{2}$ ergodic measures.

Open questions

- Closure under taking factors ?

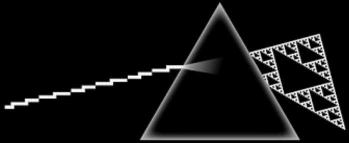
  [Y is a *factor* of X, if there is a sliding block (not necessary bijective) $\phi : X \to Y$]

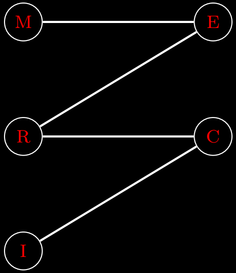
- Subgroup generated by sets of return words in an eventually dendric set ?

  [For a dendric set, $\mathcal{R}(w)$ is a basis of the free group on $\mathcal{A}$.]

- Decidability of the (eventually) dendric condition.

  [Work in progress with Revekka Kyriakoglou and Julien Leroy]

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