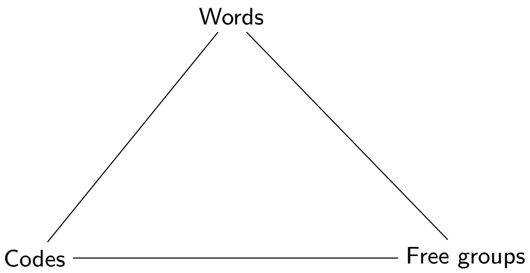
Specular sets

Francesco Dolce

Palermo, 21 dicembre 2015

Joint work with
V. Berthé, C. De Felice, V. Delecroix,
J. Leroy, D. Perrin, C. Reutenauer, G. Rindone


Words
Codes
Free groups

Words

Sturmian sets

Codes

Free groups

Bifix codes and Sturmian words

(J. Berstel, C. De Felice, D. Perrin, C. Reutenauer, G. Rindone - 2011)




Words

Sturmian sets

Codes

Free groups

Dynamical Systems



The finite index basis property

(V. Berthé, C. De Felice, F. Dolce, J. Leroy, D. Perrin, C. Reutenauer, G. Rindone - 2014)

Bifix codes and Interval Exchanges

(V. Berthé, C. De Felice, F. Dolce, J. Leroy, D. Perrin, C. Reutenauer, G. Rindone - 2014)

Words

S-adic words

Sturmian sets

Codes

Free groups

Dynamical Systems

Maximal bifix decoding

(V. Berthé, C. De Felice, F. Dolce, J. Leroy, D. Perrin, C. Reutenauer, G. Rindone - 2015)

Words

$\mathcal{S}$-adic words

Surfaces

Sturmian sets

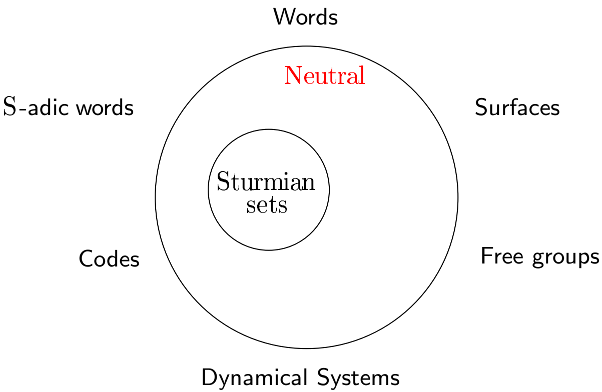
Codes

Free groups

Dynamical Systems

Return words of linear involutions and fundamental groups

(V. Berthé, V. Delecroix, F. Dolce, D. Perrin, C. Reutenauer, G. Rindone - to appear)



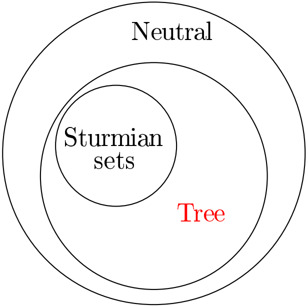
Enumeration formulæ in neutral sets

(F. Dolce, D. Perrin - DLT 2015)

Words

S-adic words

Surfaces



Codes

Free groups

Dynamical Systems

Acyclic, connected and tree sets

(V. Berthé, C. De Felice, F. Dolce, J. Leroy, D. Perrin, C. Reutenauer, G. Rindone - 2014)

On the decidability of tree condition

(F. Dolce, R. Kyriakoglou, J. Leroy - work in progress)

Introduction

Generalization of links between Sturmian sets and Free groups to general objects : *Specular sets* and *Specular groups*.

Introduction

Generalization of links between Sturmian sets and Free groups to general objects : *Specular sets* and *Specular groups*.

Introduction of new concepts : *parity* of words (*odd* and *even* words), *mixed return words*.

Introduction

Generalization of links between Sturmian sets and Free groups to general objects : *Specular sets* and *Specular groups*.

Introduction of new concepts : *parity* of words (*odd* and *even* words), *mixed return words*.

Framework allowing to handle linear involutions (generalization of interval exchange transformations).

Introduction

Generalization of links between Sturmian sets and Free groups to general objects : *Specular sets* and *Specular groups*.

Introduction of new concepts : *parity* of words (*odd* and *even* words), *mixed return words*.

Framework allowing to handle linear involutions (generalization of interval exchange transformations).

Adaptation of results holding for tree sets : "*Maximal Bifix Decoding Theorem*", "*Finite Index Basis Theorem*", "*Return Theorem*".

Outline

Outline

Motivation and Introduction

1. Specular groups
    - Groups and subgroups
    - Reduced words
    - Monoidal basis
2. Specular sets
3. Codes and subgroups

Further research directions

Given an involution $\theta : A \to A$ (possibly with some fixed point), let us define

$$G_\theta = \langle a \in A \mid a \cdot \theta(a) = 1 \text{ for every } a \in A \rangle.$$

$G_\theta = \mathbb{Z}^i * (\mathbb{Z}/2\mathbb{Z})^j$ is a *specular group* of type (i, j), and $\mathrm{Card}\,(A) = 2i + j$ is its *symmetric rank*.

Example

Let $A = \{a, b, c, d\}$ and let θ be the involution which exchanges b, d and fixes a, c, i.e.,

$$G_\theta = \langle a, b, c, d \mid a^2 = c^2 = bd = db = 1 \rangle.$$

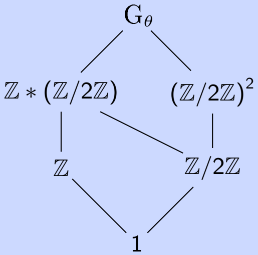
$G_\theta = \mathbb{Z} * (\mathbb{Z}/2\mathbb{Z})^2$ is a specular group of type $(1, 2)$ and symmetric rank 4.

Theorem

Any subgroup of a specular group is specular.

Example

Let $G_\theta = \mathbb{Z} * (\mathbb{Z}/2\mathbb{Z})^2$, then one has



A word is *θ-reduced* if it has no factor of the form $a\theta(a)$ for $a \in A$.

Any element of a specular group is represented by a unique reduced word.

Example

Let $\theta : b \leftrightarrow d$ fixing a, c.

The θ-reduction of the word $daaacbd$ is dac.

A word is θ*-reduced* if it has no factor of the form $a\theta(a)$ for $a \in A$.

Any element of a specular group is represented by a unique reduced word.

Example

Let $\theta : b \leftrightarrow d$ fixing a, c.

The θ-reduction of the word $d\cancel{a}\cancel{a}ac\cancel{b}\cancel{d}$ is dac.

A subset of a group G is called *symmetric* if it is closed under taking inverses (under θ).

Example

The set $X = \{a, adc, b, cba, d\}$ is symmetric, for $\theta : b \leftrightarrow d$ fixing a, c.

A subset of a group G is called *symmetric* if it is closed under taking inverses (under θ).

Example

The set $X = \{a, adc, b, cba, d\}$ is symmetric, for $\theta : b \leftrightarrow d$ fixing a, c.

A set X in a specular group G is called a *monoidal basis* of G if :

- it is symmetric ;
- the monoid that it generates is G ;
- any product $x_1 x_2 \cdots x_m$ such that $x_k x_{k+1} \neq 1$ for every k is distinct of 1.

Example

The alphabet A is a monoidal basis of G_θ.

The *symmetric rank* of a specular group is the cardinality of any monoidal basis.

Outline

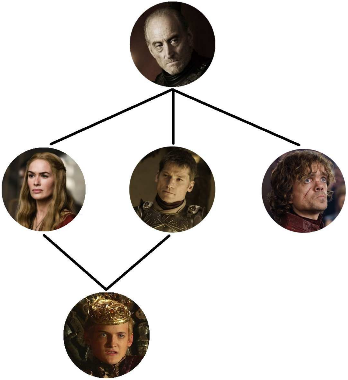
Motivation and Introduction

1. Specular groups
2. Specular sets
    - Tree sets and specular sets
    - Linear involutions and Doubling Maps
    - Even and odd words
3. Subgroup theorems

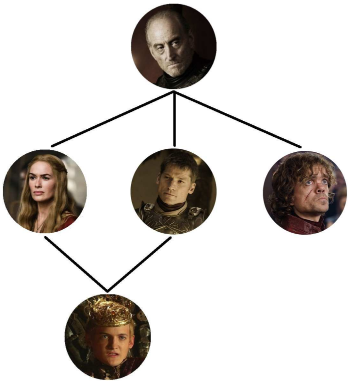
Further research directions

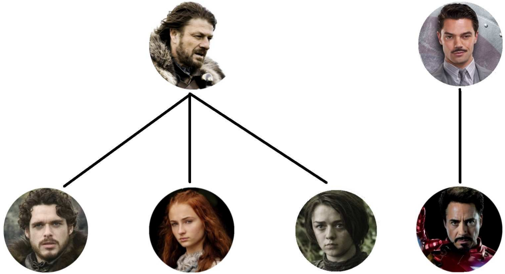
A *tree* is a graph that is both acyclic and connected.

A *tree* is a graph that is both acyclic and connected.



A *tree* is a graph that is both acyclic and connected.





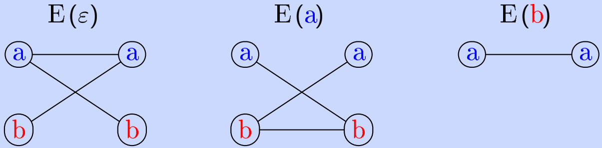
The *extension graph* of a word $w \in S$ is the undirected bipartite graph $G(w)$ with vertices the disjoint union of

$$L(w) = \{a \in A \mid aw \in S\} \quad \text{and} \quad R(w) = \{a \in A \mid wa \in S\},$$

and edges the pairs $E(w) = \{(a, b) \in A \times A \mid awb \in S\}$

Example (Fibonacci)

$S = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, bab, \ldots\}$.



A factorial and biextendable set S is called a *tree set* of *characteristic* c if for any nonempty $w \in S$, the graph $E(w)$ is a tree and if $E(\varepsilon)$ is a union of c trees.

Example

The Fibonacci set is a tree set of characteristic 1.

A factorial and biextendable set S is called a *tree set* of *characteristic* c if for any nonempty $w \in S$, the graph $E(w)$ is a tree and if $E(\varepsilon)$ is a union of c trees.

Example

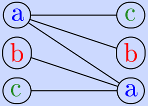
The Fibonacci set is a tree set of characteristic 1.

Proposition [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015)]

Factors of an Arnoux-Rauzy word and regular interval exchange sets are both uniformly recurrent tree sets of characteristic 1.

Example (Tribonacci)

$E(\varepsilon)$



A *specular set* on an alphabet A (w.r.t. an involution θ) is a

- biextendable and
- symmetric set
- of θ-reduced words
- which is a tree set of characteristic 2.

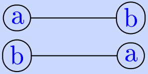
A *specular set* on an alphabet A (w.r.t. an involution θ) is a

- biextendable and
- symmetric set
- of θ-reduced words
- which is a tree set of characteristic 2.

Example

Let θ be the identity on $A = \{a, b\}$. Fac $((ab)^\omega)$ is a specular set.

$E(\varepsilon)$

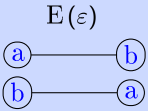


A *specular set* on an alphabet A (w.r.t. an involution θ) is a

- biextendable and
- symmetric set
- of θ-reduced words
- which is a tree set of characteristic 2.

Example

Let θ be the identity on $A = \{a, b\}$. $\mathrm{Fac}((ab)^\omega)$ is a specular set.



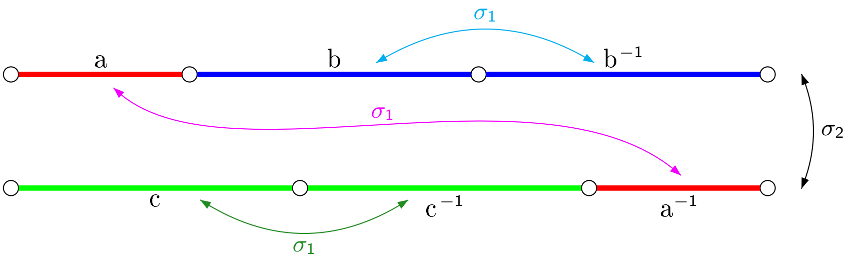
Proposition [*using* J. Cassaigne (1997)]

$p_S(0) = 1$ and $p_S(n) = n(\mathrm{Card}(A) - 2) + 2$.

Theorem

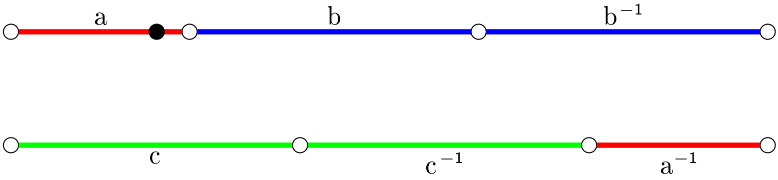
The natural coding of a linear involution without connections is a specular set.

$$T = \sigma_2 \circ \sigma_1$$



Theorem

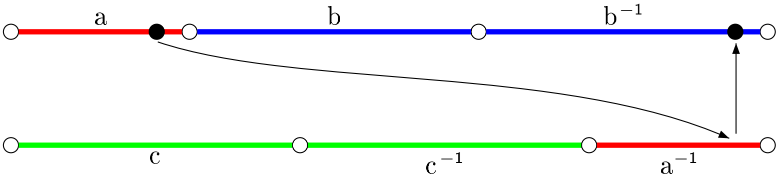
The natural coding of a linear involution without connections is a specular set.



$\Sigma_T(z) = a$

Theorem

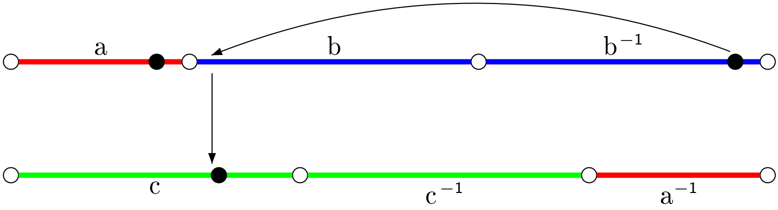
The natural coding of a linear involution without connections is a specular set.



$\Sigma T(z) = a\,b^{-1}$

Theorem

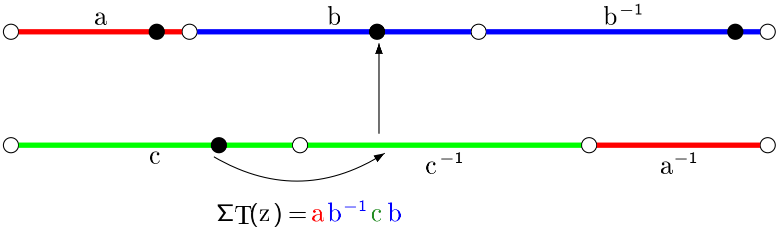
The natural coding of a linear involution without connections is a specular set.



$\Sigma T(z) = a\, b^{-1}\, c$

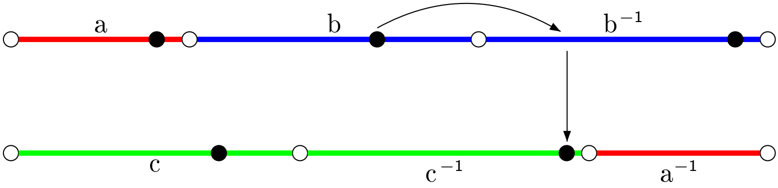
Theorem

The natural coding of a linear involution without connections is a specular set.



Theorem

The natural coding of a linear involution without connections is a specular set.



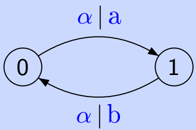
$$\Sigma T(z) = a\, b^{-1}\, c\, b\, c^{-1} \cdots$$

A *doubling transducer* is a transducer with set of states $\{0, 1\}$ such that :

1. the input automaton is a group automaton,
2. the output labels of the edges are all distinct.

Example

$\Sigma = \{\alpha\}$
$A = \{a, b\}$



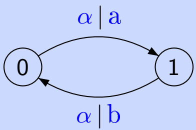
A *doubling transducer* is a transducer with set of states $\{0,1\}$ such that :

1. the input automaton is a group automaton,
2. the output labels of the edges are all distinct.

A *doubling map* is a pair $\delta = (\delta_0, \delta_1)$, where $\delta_i(u) = v$ for a path starting at the state i with input label u and output label v.

Example

$\Sigma = \{\alpha\}$
$A = \{a, b\}$



$\delta_0(\alpha^\omega) = (ab)^\omega$
$\delta_1(\alpha^\omega) = (ba)^\omega$

A *doubling transducer* is a transducer with set of states $\{0, 1\}$ such that :

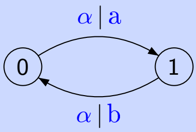
1. the input automaton is a group automaton,
2. the output labels of the edges are all distinct.

A *doubling map* is a pair $\delta = (\delta_0, \delta_1)$, where $\delta_i(u) = v$ for a path starting at the state i with input label u and output label v.

The *image* of a set T is $\delta(T) = \delta_0(T) \cup \delta_1(T)$.

Example

$\Sigma = \{\alpha\}$
$A = \{a, b\}$



$\delta_0(\alpha^\omega) = (ab)^\omega$
$\delta_1(\alpha^\omega) = (ba)^\omega$

$\delta(\alpha^\omega) = (ab)^\omega \cup (ba)^\omega$

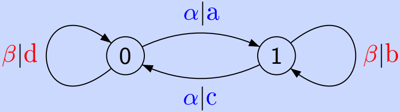
Proposition

The image of a tree set of characteristic 1 closed under reversal is a specular set.

Example (two doublings of Fibonacci on $\Sigma = \{\alpha, \beta\}$)

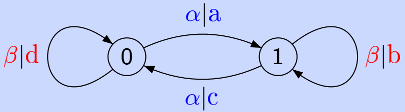
- Fac (abaababa ⋯) ∪ Fac (cdccdcdc ⋯),

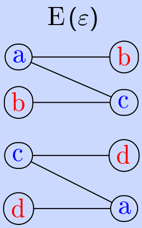
- Fac (abcabcda ⋯) ∪ Fac (cdacdabc ⋯) .



A letter is *even* if its two occurences (as a element of $\mathrm{L}(\varepsilon)$ and of $\mathrm{R}(\varepsilon)$) appear in the same tree of $\mathrm{E}(\varepsilon)$. Otherwise it is *odd*.

Example (doubling of Fibonacci)





The letters b and d are even,
while a and c are odd.

A word is *even* if it has an even number of odd letters. Otherwise it is *odd*.

Outline

A set $X \subset A^+$ of nonempty words over an alphabet A is a *bifix code* if it does not contain any proper prefix or suffix of its elements.

Example

- $\{aa, ab, ba\}$
- $\{aa, ab, bba, bbb\}$
- $\{ac, bcc, bcbca\}$
- $\{melo, pero, \underline{melo}grano\}$
- $\{mandarino, arancio, mand\underline{arancio}\}$

A set $X \subset A^+$ of nonempty words over an alphabet A is a *bifix code* if it does not contain any proper prefix or suffix of its elements.

Example

- $\{aa, ab, ba\}$
- $\{aa, ab, bba, bbb\}$
- $\{ac, bcc, bcbca\}$
- $\{melo, pero, melograno\}$
- $\{mandarino, arancio, mandarancio\}$

$X \subset S$ is S-*maximal* if it is not properly contained in a bifix code $Y \subset S$.

Example (Fibonacci)

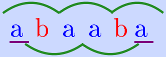
The set $X = \{aa, ab, ba\}$ is an S-maximal bifix code.
It is not an A^*-maximal bifix code, indeed $X \subset Y = X \cup \{bb\}$.

A *parse* of a word w w.r.t. a bifix code X is a triple (q, x, p) with $w = qxp$ and such that q has no suffix in X, $x \in X^*$ and p has no prefix in X.

Example

Let $X = \{aa, ab, ba\}$ and $w = abaaba$. The two possible parses of w are

- $(\varepsilon, ab{\cdot}aa{\cdot}ba, \varepsilon)$,
- $(a, ba{\cdot}ab, a)$.

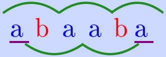


A *parse* of a word w w.r.t. a bifix code X is a triple (q, x, p) with $w = qxp$ and such that q has no suffix in X, $x \in X^*$ and p has no prefix in X.

Example

Let $X = \{aa, ab, ba\}$ and $w = abaaba$. The two possible parses of w are

- $(\varepsilon, ab{\cdot}aa{\cdot}ba, \varepsilon)$,
- $(a, ba{\cdot}ab, a)$.



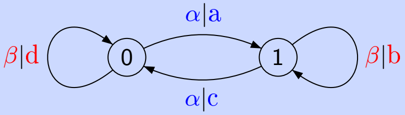
The S-*degree* of X is the maximal number of parses w.r.t. X of a word of S.

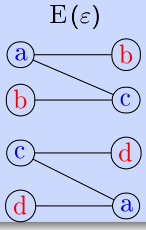
Example

- For $S =$ Fibonacci, the set $X = \{aa, ab, ba\}$ has S-degree 2;
- The set $X = S \cap A^n$ has S-degree n.

The set of even words has the form $X^* \cap S$, where $X \subset S$ is a bifix code called the *even code*.
X is the set of even words without a nonempty even prefix (or suffix).

Example (doubling of Fibonacci)





The even code is $X = \{abc, ac, b, ca, cda, d\}$.

Proposition

If S is recurrent, the even code is an S-maximal bifix code of S-degree 2.

A *coding morphism* for a (S-maximal) bifix code X is a morphism $f : B^* \to A^*$ which maps bijectively an alphabet B onto X.

The set $f^{-1}(S)$ is called a (*maximal*) *bifix decoding* of S.

A *coding morphism* for a (S-maximal) bifix code X is a morphism $f : B^* \to A^*$ which maps bijectively an alphabet B onto X.

The set $f^{-1}(S)$ is called a (*maximal*) *bifix decoding* of S.

Maximal Bifix Decoding Theorem

The decoding of a recurrent specular set by the even code is a union of two recurrent tree sets of characteristic 1.

Example (Fac $((ab)^\omega)$)

The even code is $X = \{ab, ba\}$. Let $f : \{u, v\}^* \to A^*$ be the coding morphism :

$$f : \begin{cases} u \mapsto ab \\ v \mapsto ba \end{cases}$$

Then, $f^{-1}(S) = \mathrm{Fac}(u^\omega) \cup \mathrm{Fac}(v^\omega)$.

Finite Index Basis Theorem

Let S be a recurrent specular set and $X \subset S$ a symmetric bifix code.
Then X is :
S-maximal of S-degree $d \iff$ monoidal basis of $H \leq G_\theta$, with $[G_\theta : H] = d$.

Example

- $S \cap A^d$ is a monoidal basis of $\langle A^d \rangle$.
- The even code is a monoidal basis of the *even subgroup*.

Finite Index Basis Theorem

Let S be a recurrent specular set and $X \subset S$ a symmetric bifix code.
Then X is :
S-maximal of S-degree $d \iff$ monoidal basis of $H \leq G_\theta$, with $[G_\theta : H] = d$.

Example

- $S \cap A^d$ is a monoidal basis of $\langle A^d \rangle$.
- The even code is a monoidal basis of the *even subgroup*.

The Finite Index Basis Theorem has also a converse.

Theorem

Let S be a recurrent and symmetric set of reduced words having factor complexity $p_S(n) = n\,(\text{Card}\,(A) - 2) + 2$.
If $S \cap A^n$ is a monoidal basis of $\langle A^n \rangle$ for all $n \geq 1 \implies S$ is specular.

A (*right*) *return word* to w in S is a nonempty word u such that $wu \in S \cap A^*w$, but has no internal factor equal to w.

We denote by $\mathcal{R}S(w)$ the set of return words to w in S.

Example (Fibonacci)

$\mathcal{R}S(aa) = \{baa, babaa\}$.

$$\varphi(a)^\omega = abaababaabaababaababaabaababaabaab\cdots$$

Remark.
A recurrent set S is uniformly recurrent $\iff$ $\mathcal{R}S(w)$ is finite for every $w \in S$.

Theorem [Balková, Palentová, Steiner (2008)]

Let S be a (uniformly) recurrent tree set of characteristic 1.
For every $w \in S$, the set $\mathcal{R}S(w)$ has exactly $\mathrm{Card}\,(A)$ elements.

Theorem [Balková, Palentová, Steiner (2008)]

Let S be a (uniformly) recurrent tree set of characteristic 1.
For every $w \in S$, the set $\mathcal{R}_S(w)$ has exactly $\mathrm{Card}(A)$ elements.

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

Let S be a (uniformly) recurrent tree set of characteristic 1.
For every $w \in S$, the set $\mathcal{R}_S(w)$ is a (tame) basis of the free group on A.

Theorem [Balková, Palentová, Steiner (2008)]

Let S be a (uniformly) recurrent tree set of characteristic 1.
For every $w \in S$, the set $\mathcal{R}S(w)$ has exactly $\mathrm{Card}\,(A)$ elements.

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

Let S be a (uniformly) recurrent tree set of characteristic 1.
For every $w \in S$, the set $\mathcal{R}S(w)$ is a (tame) basis of the free group on A.

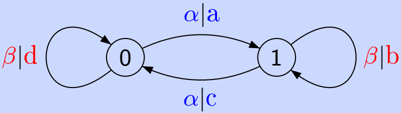
Return Theorem

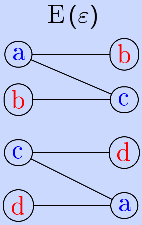
Let S be a (uniformly) recurrent specular set on the alphabet A.
For any $w \in S$, the set $\mathcal{R}S(w)$ is a monoidal basis of the even subgroup.

In particular, $\mathrm{Card}\,(\mathcal{R}S(x)) = \mathrm{Card}\,(A) - 1$.

Example (doubling of Fibonacci)

Recall that in G_θ one has $\theta : b \leftrightarrow d$ fixing a and c.



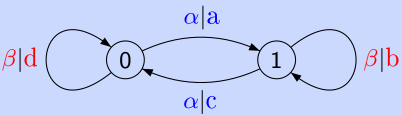


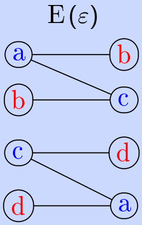
The even code is $X = \{abc, ac, b, ca, cda, d\}$,

while $\mathcal{R}S(a) = \{bc\underline{a}, bcd\underline{a}, cd\underline{a}\}$.

Example (doubling of Fibonacci)

Recall that in G_θ one has $\theta : b \leftrightarrow d$ fixing a and c.





The even code is $X = \{abc, ac, b, ca, cda, d\}$,

while $\mathcal{RS}(a) = \{bc\underline{a}, bcd\underline{a}, cd\underline{a}\}$.

One has $\langle \mathcal{RS}(a) \rangle = \langle X \rangle$, indeed :

$$\begin{cases} cda = cda & ca = (b)^{-1}(bca) \\ abc = (cda)^{-1} & ac = (ca)^{-1} \\ b = (bcda)(abc) & d = b^{-1} \end{cases}$$

THESE ~~AREN'T~~ THE

FUTURE RESEARCH DIRECTIONS

YOU'RE LOOKING FOR

- Recurrence and uniformly recurrence in tree sets.
- Bifix decoding for general bifix codes.
- Decidability of the tree condition.
- Connection with G-full (or G-rich) words.
- Generalization towards larger classes of groups (virtually free).
- Profinite monoids and profinite groups.

GRAZIE