Palindromes and Tree Sets

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LaCIM UQÀM

Secondo Incontro di Combinatoria delle Parole

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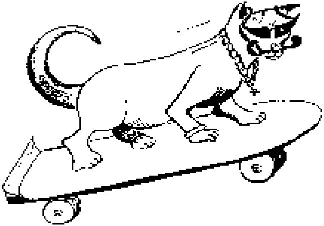
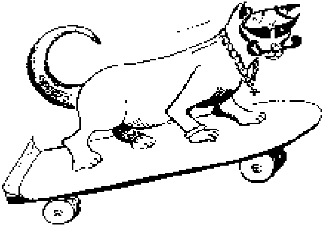


GOFLOWOLFOG



GOFLOWOLFOG

“ You can summon him by trying to take on his characteristics - relaxing, fantasising that you’re ‘cool’, and letting go of your frustration momentarily. Visualise him zipping along on his skateboard, accompanied by a slight breeze and his Mantra : ‘Neeeoooow’. ”

[philhine.org.uk]

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- E Le Mie Sei Mele ?

Full words

Theorem [X. Droubay, J. Justin, G. Pirillo (2001)]

A word of length n has at most $n+1$ palindrome factors

A word with maximal number of palindromes is *rich* (or *full*).

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Example

- ANANAS, BANANA, LAMPONE, SUSINA are rich.
- ALBICOCCA, ANGURIA, FRAGOLA, MELONE are not rich.

$$|\textsc{susina}| = 6 \quad \text{and} \quad \text{Card}\left(\{\varepsilon, \textsc{a}, \textsc{i}, \textsc{n}, \textsc{s}, \textsc{u}, \textsc{sus}\}\right) = 7$$

$$|\textsc{albicocca}| = 9 \quad \text{and} \quad \text{Card}\left(\{\varepsilon, \textsc{a}, \textsc{b}, \textsc{c}, \textsc{i}, \textsc{l}, \textsc{o}, \textsc{cc}, \textsc{coc}\}\right) = 9$$

Full words

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A word of length n has at most $n+1$ palindrome factors

A word with maximal number of palindromes is *rich* (or *full*).
A factorial set is *rich* if all its elements are rich.

Example (Fibonacci)

Let S be the set of factors of the fixed-point $\varphi^\omega(a)$ of

$$\varphi : a \mapsto ab, \quad b \mapsto a.$$

Every word $w \in S$ is rich. For instance,

$$\mathrm{Pal}(abaab) = \{\varepsilon, a, b, aa, aba, baab\}.$$

Arnoux-Rauzy sets

Definition

An *Arnoux-Rauzy* set is a factorial set closed under reversal with $p_n = (\text{Card}\,(A) - 1)n + 1$ having a unique right special factor for each length.

Examples

- Fibonacci : factors of the fixed-point $\varphi^\omega(a)$, where $\varphi : \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$.
- Tribonacci : factors of the fixed-point $\psi^\omega(a)$, where $\psi : \begin{cases} a \mapsto ab \\ b \mapsto ac \\ c \mapsto a \end{cases}$.

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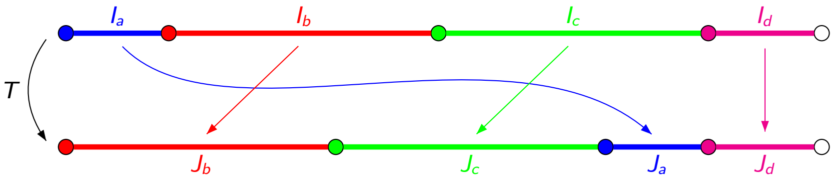
Theorem [X. Droubay, J. Justin, G. Pirillo (2001)]

Arnoux-Rauzy sets are rich.

Interval exchanges

Let $(I_\alpha)_{\alpha\in A}$ and $(J_\alpha)_{\alpha\in A}$ be two partitions of $[0,1[$.
An *interval exchange transformation* (IET) is a map $T : [0,1[\to [0,1[$ defined by

$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha.$$



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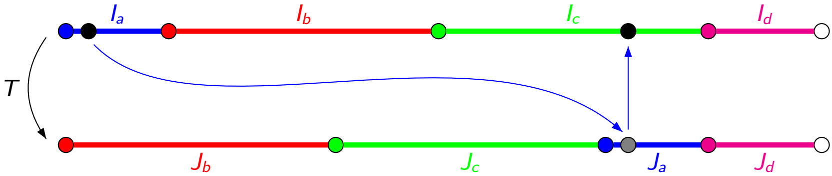

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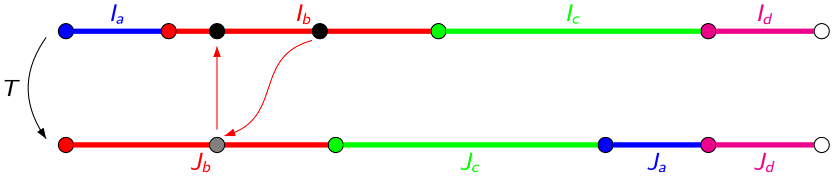

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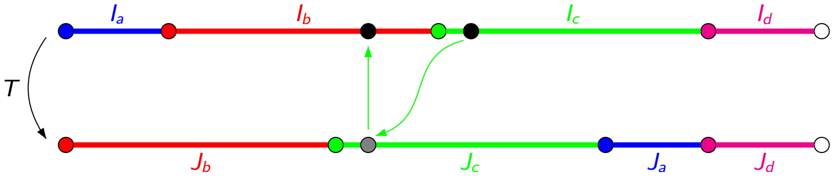

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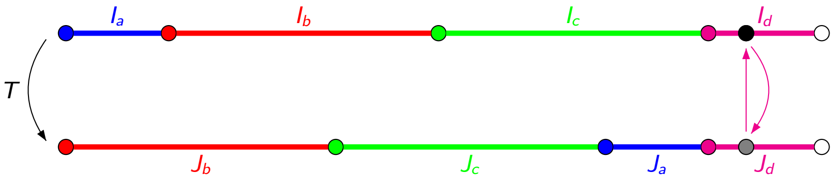

Interval exchanges

T is said to be *minimal* if for any point $z \in [0,1[$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[0,1[$.

T is said *regular* if the orbits of the separation points $\neq 0$ are infinite and disjoint.

Theorem [M. Keane (1975)]

A regular interval exchange transformation is minimal.

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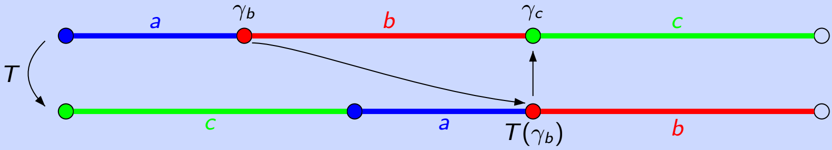
Example (the converse is not true)

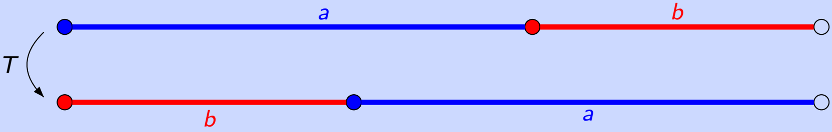


Interval exchanges

The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in A^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha .$$

Example (Fibonacci, $z = (3-\sqrt{5})/2$)



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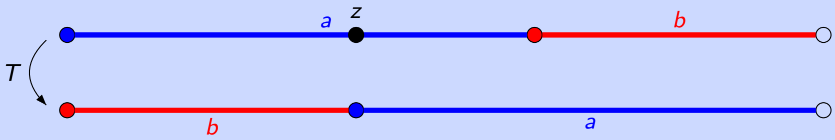

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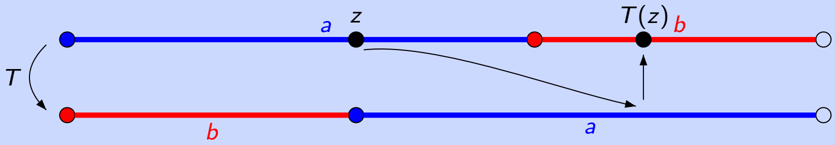

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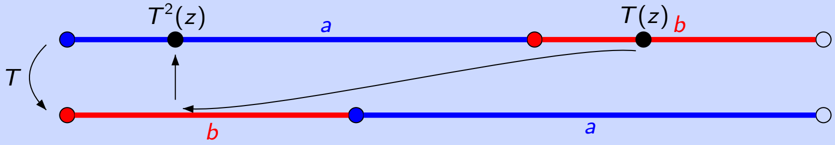

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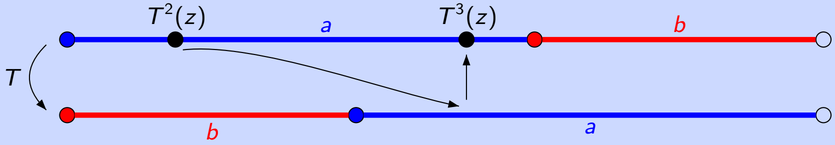

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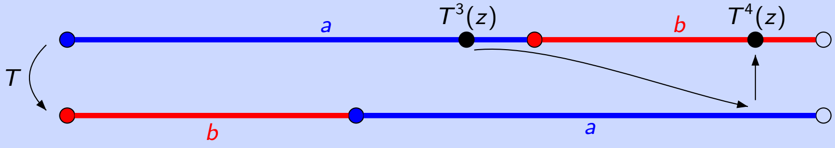

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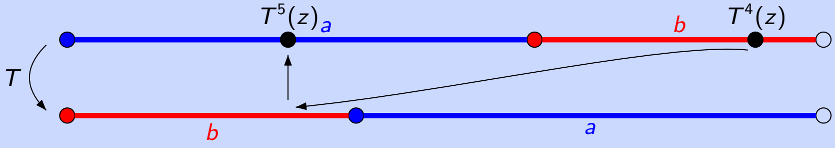

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The set $\mathcal{L}(T) = \bigcup_{z \in [0,1[} \mathrm{Fac}(\Sigma_T(z))$ is said a (*minimal*, *regular*) *interval exchange set*.

Remark. If T is minimal, $\mathrm{Fac}(\Sigma_T(z))$ does not depend on the point z.

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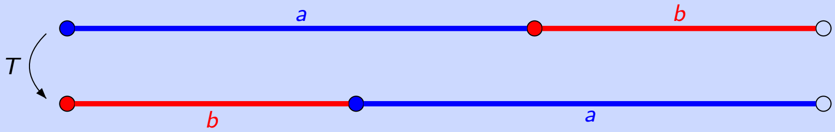

$$\mathcal{L}(T) = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, \dots\}$$

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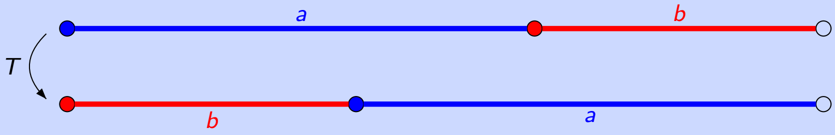


$$\mathcal{L}(T) = \left\{ \varepsilon, a, b, aa, ab, ba, aab, aba, baa, \dots \right\}$$

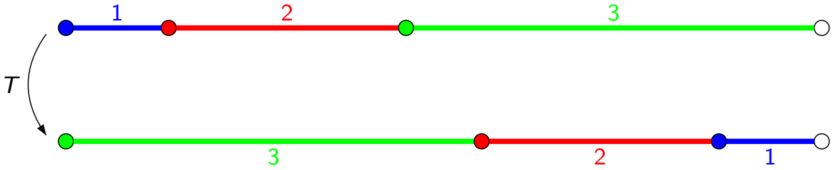
Proposition

Regular interval exchange sets have factor complexity $p_n = (\mathrm{Card}(A) - 1)n + 1$.

Interval exchanges

Theorem [P. Baláži, Z. Masáková, E. Pelantová (2007)]

Regular interval exchange sets closed under reverse are rich.



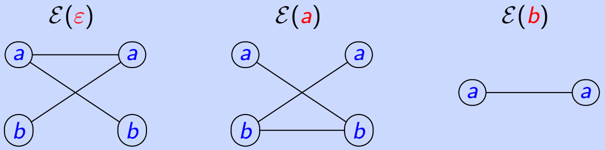
T closed under reverse $\iff$ $\pi = (n \ \ n-1 \ \ \cdots \ \ 2 \ \ 1)$

Extension graphs

The *extension graph* of a word $w \in S$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$
\begin{aligned}
L(w) &= \{a \in A \,|\, aw \in S\}, \\
R(w) &= \{a \in A \,|\, wa \in S\}, \\
B(w) &= \{(a, b) \in A \,|\, awb \in S.\}
\end{aligned}
$$

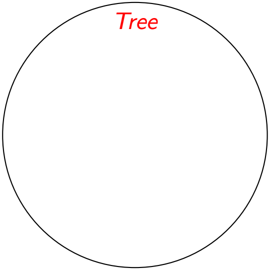
Example (Fibonacci, $S = \{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, bab, \ldots\}$)



Tree sets

Definition

A factorial set S is called a *tree set* (of characteristic 1) if $\mathcal{E}(w)$ is a tree for any $w \in S$.



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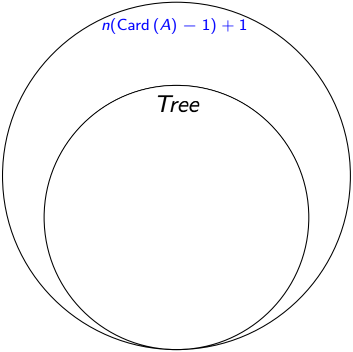

[*using* J. Cassaigne : "**Complexité et facteurs spéciaux**" (1997).]

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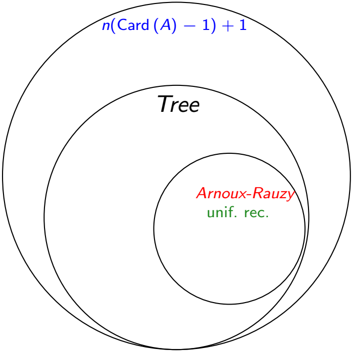

[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone : "**Acyclic, connected and tree sets**" (2014).]

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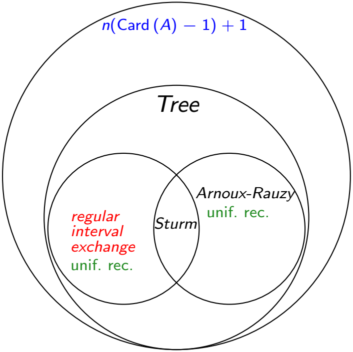

[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone : "**Bifix codes and interval exchanges**" (2015).]

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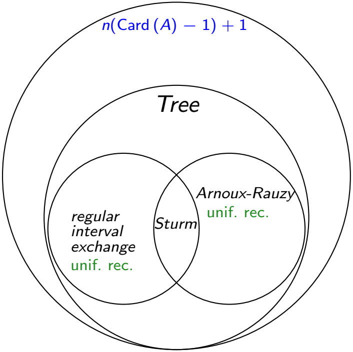


Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2016)]

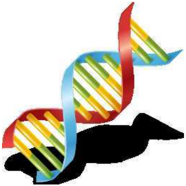
A (uniformly) recurrent tree set closed under reversal is rich.

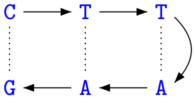
σ-*palindromes*

Let σ be an antimorphism.
A word w is a σ-*palindrome* if $w = \sigma(w)$.

Example

Let $\sigma : \mathtt{A} \leftrightarrow \mathtt{T},\ \mathtt{C} \leftrightarrow \mathtt{G}$.
The word CTTAAG is a σ-palindrome.





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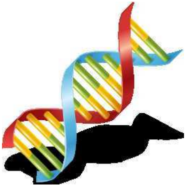

Theorem [Š. Starosta (2011)]

Let $\gamma_\sigma(w)$ be the number of transpositions of σ affecting w. Then

$$\text{Card}\,(\text{Pal}_\sigma(w)) \leq |w| + 1 - \gamma_\sigma(w)$$

A word (resp. set) is *σ-rich* if the equality holds (resp. for all its elements).

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Example

Let $\sigma : \mathtt{A} \leftrightarrow \mathtt{L},\ \mathtt{B} \leftrightarrow \mathtt{E},\ \mathtt{I} \leftrightarrow \mathtt{R}$ and $\tau = \mathrm{id}$.

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$$\begin{aligned} \mathrm{Card}\,(\mathrm{Pal}_\sigma(\mathtt{GABRIELE})) &= \mathrm{Card}\,(\{\varepsilon,\ \mathtt{G},\ \mathtt{RI},\ \mathtt{BRIE},\ \mathtt{ABRIEL}\}) \\ &= 5 < 6 = 8 + 1 - 3 \end{aligned}$$

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&= 5 < 6 = 8 + 1 - 3 \\
\text{Card}\,(\text{Pal}_\tau(\texttt{CLELIA})) &= \text{Card}\,(\{\varepsilon,\ \texttt{C},\ \texttt{L},\ \texttt{E},\ \texttt{I},\ \texttt{A},\ \texttt{LEL}\}) \\
&= 5 = 6 + 1 - 2
\end{aligned}$$

G-palindromes

Let G be a group containing at least one antimorphism.
A word w is a *G-palindrome* if there exists a nontrivial $g \in G$ s.t. $w = g(w)$.

Example

Let $G = \langle \sigma, \tau \rangle$ with $\sigma : \mathtt{A} \leftrightarrow \mathtt{X},\ \mathtt{D} \leftrightarrow \mathtt{E},\ \mathtt{M} \leftrightarrow \mathtt{Q},\ \mathtt{O} \leftrightarrow \mathtt{U}$ and $\tau : \mathtt{A} \leftrightarrow \mathtt{O},\ \mathtt{D} \leftrightarrow \mathtt{L},\ \mathtt{U} \leftrightarrow \mathtt{X}$.

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- `ALDO`, fixed by τ,

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Example

Let $G = \langle \sigma, \tau \rangle$ with $\sigma : \mathtt{A} \leftrightarrow \mathtt{X},\ \mathtt{D} \leftrightarrow \mathtt{E},\ \mathtt{M} \leftrightarrow \mathtt{Q},\ \mathtt{O} \leftrightarrow \mathtt{U}$ and $\tau : \mathtt{A} \leftrightarrow \mathtt{O},\ \mathtt{D} \leftrightarrow \mathtt{L},\ \mathtt{U} \leftrightarrow \mathtt{X}$.

The following are G-palindromes :

- `DOMINIQUE`, fixed by σ,
- `ALDO`, fixed by τ,
- `ANTONIO`, fixed by $\tau\sigma\tau\sigma$.

G-palindromes

Let G be a group containing at least one antimorphism.
A word w is a *G-palindrome* if there exists a nontrivial $g \in G$ s.t. $w = g(w)$.

Example

Let $G = \langle \sigma, \tau \rangle$ with $\sigma : \mathtt{A} \leftrightarrow \mathtt{X},\ \mathtt{D} \leftrightarrow \mathtt{E},\ \mathtt{M} \leftrightarrow \mathtt{Q},\ \mathtt{O} \leftrightarrow \mathtt{U}$ and $\tau : \mathtt{A} \leftrightarrow \mathtt{O},\ \mathtt{D} \leftrightarrow \mathtt{L},\ \mathtt{U} \leftrightarrow \mathtt{X}$.

The following are G-palindromes :

- DOMINIQUE, fixed by σ,
- ALDO, fixed by τ,
- ANTONIO, fixed by $\tau\sigma\tau\sigma$.

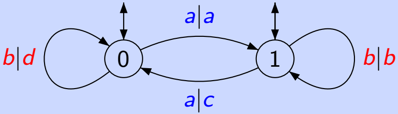
A word (set) is *G-rich* if "the number of G-palindromes is maximal".

G-*palindromes*

Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2016)]

Let S be a tree set closed under reversal.
The set obtained from S using a *doubling transducer* is G-rich, with $G \simeq (\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/2\mathbb{Z})$.

Example (doubling of Fibonacci)



$$aba \longrightarrow abc, cda$$

GRAZIEIZARG