Playing with games and words

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Nim Game

(Drinking version)

Initial position: Arbitrary number of piles, of arbitrary sizes, of glasses of wine.

Rules:

i) At each turn a player drinks a positive number of glasses from one pile.

Winner: Who drinks the last glass.

Nim Game

Using some math

Let us denote by $(a_1, a_2, \ldots, a_n)$ be a game position. A position is in $\mathcal{P}$ if there exists a *winning strategy* for the second player. Otherwise it is in $\mathcal{N}$.

- $(0, 0, \ldots, 0) \in \mathcal{P}$;

  (*The last player wins*)
- $(a_1, a_2, \ldots, a_n) \in \mathcal{P} \quad \Rightarrow \quad Nim(a_1, a_2, \ldots, a_n) \subseteq \mathcal{N}$;

  (*Any move from $\mathcal{P}$ leads to $\mathcal{N}$*)
- $(a_1, a_2, \ldots, a_n) \in \mathcal{N} \quad \Rightarrow \quad Nim(a_1, a_2, \ldots, a_n) \cap \mathcal{P} \neq \emptyset$.

  (*From any position in $\mathcal{N}$ there exists a move leading to a position in $\mathcal{P}$*)

Nim Game

Using some math

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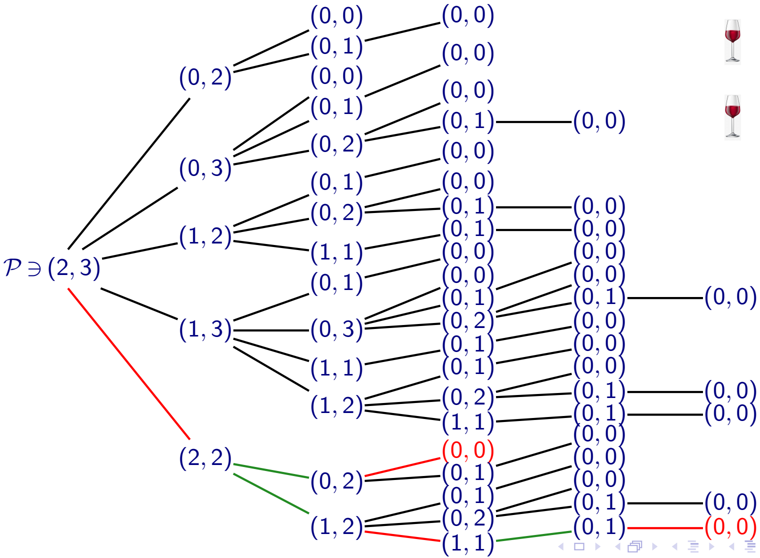
  (*From any position in $\mathcal{N}$ there exists a move leading to a position in $\mathcal{P}$*)

Thus $(a_1, a_2, \dots, a_n) \in \mathcal{P}$ if $\underline{\forall\exists\forall\exists\cdots\forall\exists}$ moves s.t. we obtain $(0, 0, \dots, 0)$.

Nim Game

∃∀∃∀∃



$\mathcal{P} \ni (2,3)$
(0, 2)
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Nim Game

Using more math

Question: How to determine whether a position is in $\mathcal{P}$?

Theorem [C. Bouton (1904)]

A position $(a_1, a_2, \ldots, a_n)$ is in $\mathcal{P}$ if its *Nim-sum* is 0.

$2 \oplus 4 \oplus 6 = 0$

	1	0
1	0	0
1	1	0
0	0	0

$3 \oplus 6 \oplus 8 \neq 0$

		1	1
	1	1	0
1	0	0	0
1	1	0	1

Wythoff's Game

A modification of Nim Game

Initial position: Two piles, of arbitrary sizes, of glasses of wines.
Rules: At each turn a player drinks either

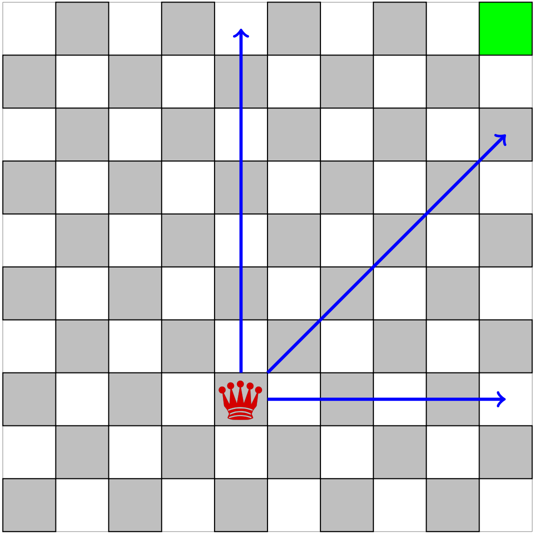
i) a positive number of glasses from one pile, or

ii) a positive equal number of glasses from both piles.

Winner: Who drinks the last glass of wine.

Wythoff's Game

Playing chess



Wythoff's Game

Safe positions

Question: How to compute the set $\mathcal{P}$?

- $(0,0) \in \mathcal{P}$ but $(n,n) \in \mathcal{N}$ for every $n > 0$;
- if $(a,b) \in \mathcal{P}$ then $(a+k, b+k) \in \mathcal{N}$ for every $k > 0$;
- $(a,b) \in \mathcal{P}$ iff $(b,a) \in \mathcal{P}$ [*Thus, wlog,* $0 \leq a \leq b$].

Wythoff's Game

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- $(a,b) \in \mathcal{P}$ iff $(b,a) \in \mathcal{P}$ [*Thus, wlog,* $0 \leq a \leq b$].

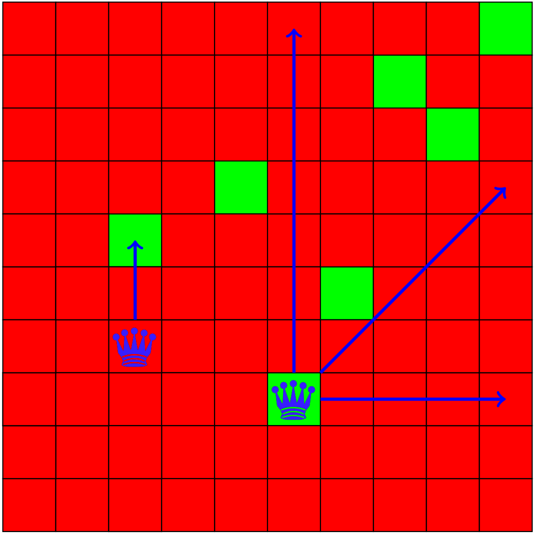
Theorem [W. Wythoff (1907)]

The set $\mathcal{P}$ is defined by the positions $\{(a_n, b_n)\}_{n\in\mathbb{N}}$, where $(a_0, b_0) = (0,0)$ and

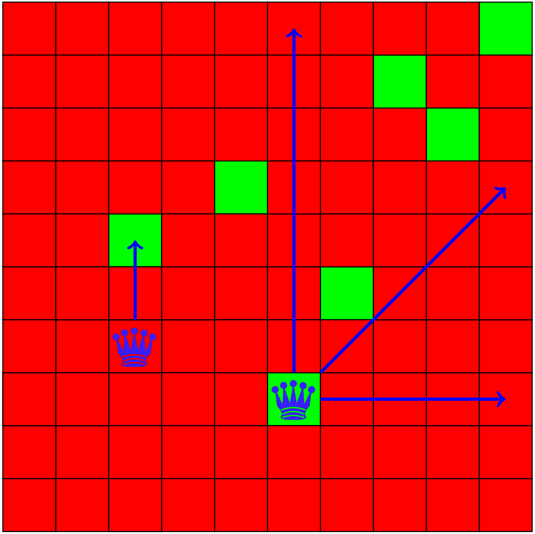
$$\begin{cases} a_n = \mathrm{Mex}\left(\{a_i, b_i \mid 0 \leq i < n\}\right), \\ b_n = a_n + n. \end{cases}$$

Thus $\mathcal{P}$ contains: $(0,0)$, $(1,2)$, $(3,5)$, $(4,7)$, $(6,10)$, $\ldots$.

On the chessboard again



On the chessboard again



Problem: Is it possible to compute the sequence in polyomial time?

Theorem [W. Wythoff (1907)]

The set $\mathcal{P}$ is defined by the positions $\{(a_n, b_n)\}_{n\in\mathbb{N}}$, where

$$a_n = \lfloor n\tau \rfloor \qquad b_n = \lfloor n\tau^2 \rfloor$$

where $\tau = \frac{1+\sqrt{5}}{2}$ (and thus $\tau^2 = \frac{3+\sqrt{5}}{2}$).

Wythoff's Game

Algebraic characterisation

Theorem [W. Wythoff (1907)]

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where $\tau = \frac{1+\sqrt{5}}{2}$ (and thus $\tau^2 = \frac{3+\sqrt{5}}{2}$).

Proof.

→ Easy to see that $b_n - a_n = n$.

→ Prove that every positive integer appears exactly once is a bit more complicated...

- For every irrational α the set of infinite pairs $\{\lfloor n\alpha \rfloor, \lfloor n\frac{\alpha}{\alpha-1} \rfloor\}_{n\in\mathbb{N}}$ is a *(eventual) covering family*, i.e., it covers $\mathbb{Z}$.
- $\alpha - \frac{\alpha}{\alpha-1} = 1 \quad \Leftrightarrow \quad \alpha = \tau$

Fibonacci word

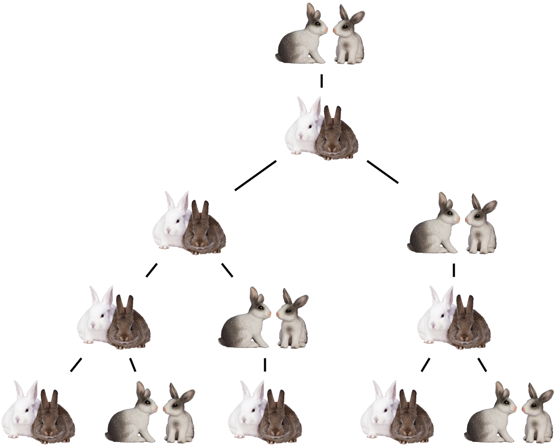
$$\mathbf{f} = \mathtt{abaababaabaababaaba}\cdots$$

$$\mathbf{f} = \lim_{n\to\infty} \varphi^n(\mathtt{a}) \qquad \text{where} \qquad \varphi : \begin{cases} \mathtt{a} \mapsto \mathtt{ab} \\ \mathtt{b} \mapsto \mathtt{a} \end{cases}$$

The length of prefixes $|\varphi^n(\mathtt{a})|_n = (1,)1, 2, 3, 5, 8, \ldots$ are the *Fibonacci numbers*.

Fibonacci numbers

and bunnies



1

1

2

3

5

Fibonacci word

$$\mathbf{f} = \texttt{abaababaabaababaaba}\cdots$$

Let a_n denote the n^{th} occurrence of a and b_n denote the n^{th} occurrence of b.

$$(a_n)_{n\geq 1} = 1, 3, 4, 6, 8, 9, \ldots \qquad (b_n)_{n\geq 1} = 2, 5, 7, 10, 13, 15, \ldots$$

The *golden ration* τ is exactly the *frequence* of a in **f** (and τ^2 the frequence of b).

Fibonacci word

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Theorem [Duchêne, Rigo (2008)]

Let $a_0 = b_0 = 0$. The sequence $(a_n, b_n)_{n\in\mathbb{N}}$ is the Wythoff's sequence.

Fibonacci word

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Theorem [Duchêne, Rigo (2008)]

Let $a_0 = b_0 = 0$. The sequence $(a_n, b_n)_{n\in\mathbb{N}}$ is the Wythoff's sequence.

Proof.

→ All b are created by a, the gaps are filled with a and $a_n = \mathrm{Mex}(\{a_i, b_i \mid 0 \leq i < n\})$.

→ Since **f** starts with ab, then $b_1 = 2 = a_1 + 1$;

Let us suppose that $b_{n-1} = a_{n-1} + n - 1$.

- Since $\varphi(\mathtt{aa}) = \mathtt{abab}$, if $a_n - a_{n-1} = 1$ then $b_n - b_{n-1} = 2$;
- Since $\varphi(\mathtt{aba}) = \mathtt{abaab}$, if $a_n - a_{n-1} = 2$ then $b_n - b_{n-1} = 3$;

In both case $b_n = a_n + n$.

Sturmian words

Definition

An infinite word $\mathbf{w}$ is *Sturmian* if it has $n+1$ distinct factors of length n for every $n \geq 0$.

Example (Fibonacci)

$$\mathbf{f} = \mathtt{abaababaabaababa}\cdots$$

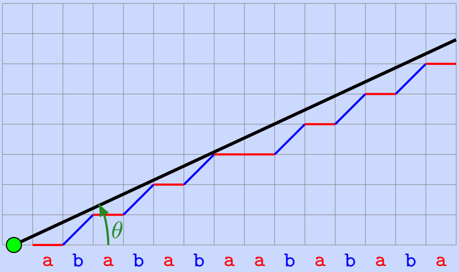
$$\mathcal{L}(\mathbf{f}) = \{\underbrace{\varepsilon}_{1}, \underbrace{\mathtt{a}, \mathtt{b}}_{2}, \underbrace{\mathtt{aa}, \mathtt{ab}, \mathtt{ba}}_{3}, \underbrace{\mathtt{aab}, \mathtt{aba}, \mathtt{baa}, \mathtt{bab}}_{4}, \underbrace{\mathtt{aaba}, \mathtt{abaa}, \mathtt{abab}, \mathtt{baab}, \mathtt{baba}}_{5}, \ldots\}$$

Sturmian words

Definition

An infinite word $\mathbf{w}$ is *Sturmian* if it has $n+1$ distinct factors of length n for every $n \geq 0$.
A Sturmian word can also be represented geometrically.

Example (Fibonacci)



$$\begin{cases} \mathtt{a} \text{ if } \lfloor (n+1)\theta + \rho \rfloor - \lfloor n\theta + \rho \rfloor = 0 \\ \mathtt{b} \text{ if } \lfloor (n+1)\theta + \rho \rfloor - \lfloor n\theta + \rho \rfloor = 1 \end{cases}$$

Modified Wythoff's Games?

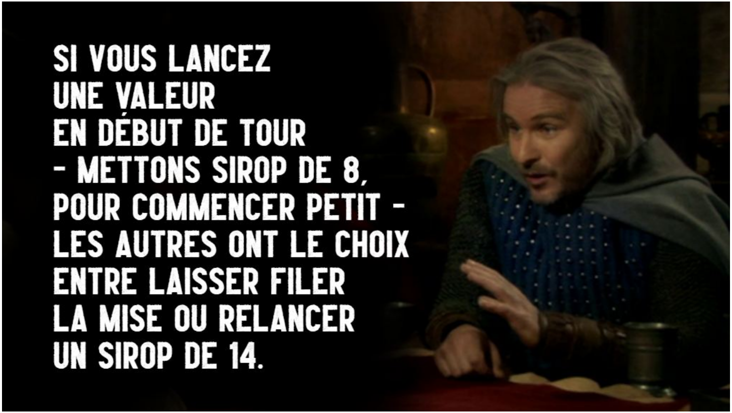
always with two piles

Question: Let **x** be a Sturmian word (maybe with some extra hypotheses).
Is it possible to define a new game (*similar rules as Wythoff's one*) such that

$$(A, B) \in \mathcal{P} \quad \text{if and only if} \quad A = a_n \text{ and } B = b_n$$

with a_n (resp. b_n) the n^{th} occurrence of a (resp. of b) in **x**?

More complicated games



Modified Wythoff's Game

Tribonacci game

Initial position: Three piles, of arbitrary sizes, of glasses of wine.
Rules: At each turn a player drinks either

i) a positive number of glasses from one pile; or

ii) a positive number α, β and γ of glasses from the first, second and third pile whenever $2\max\{\alpha, \beta, \gamma\} \leq \alpha + \beta + \gamma$; or

iii) the same positive number α of glasses from two piles and β from the other pile whenever $\beta > 2\alpha > 0$ and $a' < c' < b'$, with (a, b, c) the original position and (a', b', c') the new one.

Winner: Who drinks the last glass of wine.

Arnoux-Rauzy words

Definition

An infinite word $\mathbf{w}$ over an alphabet of k letters is an *Arnoux-Rauzy word* if

1. it has $(k-1)n+1$ distinct factors of length n for every $n \geq 0$;
2. for each lenght only one factor is right special; and
3. its set of factors is closed under reversal.

Example (Tribonacci: $\psi : \mathtt{a} \mapsto \mathtt{ab},\ \mathtt{b} \mapsto \mathtt{ac},\ \mathtt{c} \mapsto \mathtt{a}$)

$$\mathbf{t} = \psi^{\omega}(\mathtt{a}) = \mathtt{abacabaabacababacabaabaca}\cdots$$

$$\mathcal{L}(\mathbf{t}) = \{\underbrace{\varepsilon}_{1}, \underbrace{\mathtt{a}, \mathtt{b}, \mathtt{c}}_{3}, \underbrace{\mathtt{aa}, \mathtt{ab}, \mathtt{ac}, \mathtt{ba}, \mathtt{ca}}_{5}, \underbrace{\mathtt{aab}, \mathtt{aba}, \mathtt{aca}, \mathtt{baa}, \mathtt{bab}, \mathtt{bac}, \mathtt{cab}}_{7}, \ldots\}$$

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Tribonacci game and Tribonacci word

$$\mathbf{t} = \mathtt{abacabaabacababacabaabaca}\cdots$$

Let a_n, b_n and c_n denote the n^{th} occurrences of a, b and c in $\mathbf{t}$ respectively.

$(a_n)_n = 1, 3, 4, 7, 8, \ldots$ $(b_n)_n = 2, 6, 9, 13, 15, \ldots$ $(c_n)_n = 4, 11, 17, 24, 28, \ldots$

Theorem [Duchêne, Rigo (2008)]

The set $\{(a_n, b_n, c_n) \mid n \geq 1\}$ is set of $\mathcal{P}$-positions of the Tribonacci game.

Proof. (idea)

$$\begin{cases} a_n = \mathrm{Mex}\left(\{a_i, b_i, c_i \mid 0 \leq i < n\}\right), \\ b_n = a_n + \mathrm{Mex}\left(b_i - a_i,\ c_i - b_i \mid 0 \leq i < n\right). \\ c_n = a_n + b_n + n \end{cases}$$

Modified Wythoff's Games?

on two or more piles

Question: Let **x** be an Arnoux-Rauzy word.
Is it possible to define a new game (*similar rules as Wythoff's one*) such that

$$(A, B, C) \in \mathcal{P} \quad \text{if and only if} \quad A = a_n, \; B = b_n \text{ and } C = c_n$$

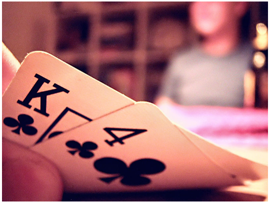
with a_n (resp. b_n, c_n) the n^{th} occurrence of a (resp. b, c) in **x**?

Different types of games

(Im)perfect information

Definition

A (sequencial) game has *perfect information* if each player knows all the previous configurations (initial configuration, moves of every players).
Whenever some configuration is hidden, the game has *imperfect* information.

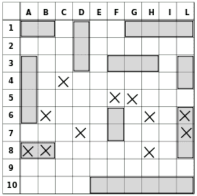


Different types of games

(In)complete information

Definition

A (sequencial) game has *complete information* if each player knows the strategies of the other player (rules of the game, goals, payoff, etc.)
Whenever players don't have full information about their opponents' strategies, the game has *incomplete information*.



Different types of games

A game with complete information may or may not have perfect information, and vice versa.
However...

Theorem [J.C. Harsanyi (1967)]

Every game with incomplete information can be modified to a game with complete but imperfect information.

Modified Nim or Wythoff's Games?

Question: Is it possible to define (and to find a winning strategy) a variation of the Nim or Wythoff's Game with imperfect (or maybe inncomplete) information?