Return words and palindromes in specular sets

Francesco Dolce

UNIVERSITÉ PARIS-EST MARNE-LA-VALLÉE

Laboratoire d'informatique Gaspard - Monge

RDMath IdF
Domaine d'Intérêt Majeur (DIM) en Mathématiques

îledeFrance

Praha, 24th May 2016

based on a joint work with

V. Berthé, C. De Felice, V. Delecroix,
J. Leroy, D. Perrin, C. Reutenauer, G. Rindone

- return words

- return words

- palindromes

Outline

Introduction

1. Specular sets

2. Return words

3. Palindromes

Conclusions

The *extension graph* of a word $w \in S$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

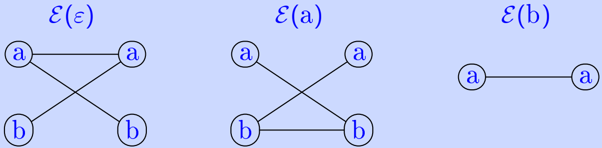
$$\begin{aligned} L(w) &= \{a \in A \mid aw \in S\}, \\ R(w) &= \{a \in A \mid wa \in S\}, \\ B(w) &= \{(a,b) \in A \times A \mid awb \in S\}. \end{aligned}$$

The *extension graph* of a word $w \in S$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{a \in A \mid aw \in S\}, \\ R(w) &= \{a \in A \mid wa \in S\}, \\ B(w) &= \{(a,b) \in A \times A \mid awb \in S\}. \end{aligned}$$

Example (Fibonacci)

$S = \{\varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab, \ldots\}.$

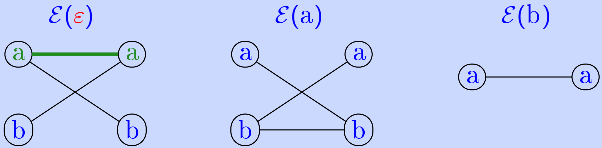


The *extension graph* of a word $w \in S$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{a \in A \mid aw \in S\}, \\ R(w) &= \{a \in A \mid wa \in S\}, \\ B(w) &= \{(a,b) \in A \times A \mid awb \in S\}. \end{aligned}$$

Example (Fibonacci)

$S = \{\varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab, \ldots\}.$

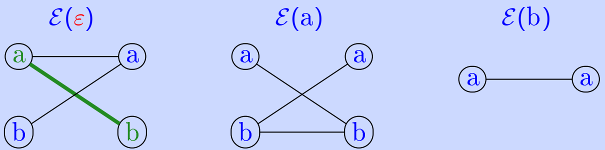


The *extension graph* of a word $w \in S$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{a \in A \mid aw \in S\}, \\ R(w) &= \{a \in A \mid wa \in S\}, \\ B(w) &= \{(a,b) \in A \times A \mid awb \in S\}. \end{aligned}$$

Example (Fibonacci)

$S = \{\varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab, \ldots\}$.

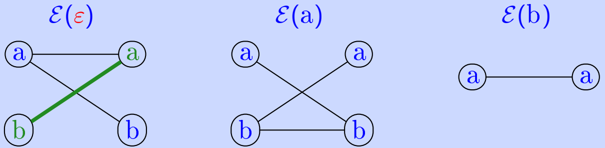


The *extension graph* of a word $w \in S$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{a \in A \mid aw \in S\}, \\ R(w) &= \{a \in A \mid wa \in S\}, \\ B(w) &= \{(a,b) \in A \times A \mid awb \in S\}. \end{aligned}$$

Example (Fibonacci)

$S = \{\varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab, \ldots\}$.

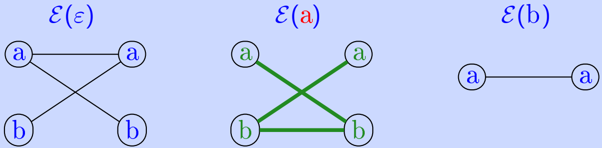


The *extension graph* of a word $w \in S$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{a \in A \mid aw \in S\}, \\ R(w) &= \{a \in A \mid wa \in S\}, \\ B(w) &= \{(a,b) \in A \times A \mid awb \in S\}. \end{aligned}$$

Example (Fibonacci)

$S = \{\varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab, \ldots\}$.

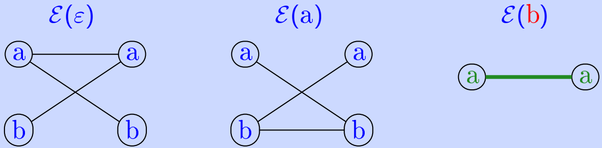


The *extension graph* of a word $w \in S$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{a \in A \mid aw \in S\}, \\ R(w) &= \{a \in A \mid wa \in S\}, \\ B(w) &= \{(a,b) \in A \times A \mid awb \in S\}. \end{aligned}$$

Example (Fibonacci)

$S = \{\varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab, \ldots\}.$



A factorial set S is called a *tree set* of *characteristic* c if $\mathcal{E}(w)$ is a tree for any nonempty $w \in S$, and $\mathcal{E}(\varepsilon)$ is a union of c trees.

A factorial set S is called a *tree set* of *characteristic* c if $\mathcal{E}(w)$ is a tree for any nonempty $w \in S$, and $\mathcal{E}(\varepsilon)$ is a union of c trees.

Theorem

Families of (uniformly) recurrent tree sets of characteristic 1 :

- Factors of Arnoux-Rauzy (*Sturmian*) words ;

  [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014)]

- Natural coding of regular interval exchanges.

  [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

A factorial set S is called a *tree set* of *characteristic* c if $\mathcal{E}(w)$ is a tree for any nonempty $w \in S$, and $\mathcal{E}(\varepsilon)$ is a union of c trees.

Theorem

Families of (uniformly) recurrent tree sets of characteristic 1 :

- Factors of Arnoux-Rauzy (*Sturmian*) words ;

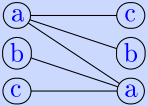
  [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014)]

- Natural coding of regular interval exchanges.

  [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

Example (Tribonacci)

$\mathcal{E}(\varepsilon)$



Let $\theta : \mathrm{A} \to \mathrm{A}$ be an involution (possibly with some fixed point).

Let $\theta : A \to A$ be an involution (possibly with some fixed point).

A word is *θ-reduced* if it has no factor of the form $a\theta(a)$ for $a \in A$.

Example

Let $\theta : a \mapsto a,\ b \mapsto d,\ c \mapsto c,\ d \mapsto b$.

The θ-reduction of the word $d\,aaacdb$ is dac.

Let $\theta : \mathrm{A} \to \mathrm{A}$ be an involution (possibly with some fixed point).

A word is *θ-reduced* if it has no factor of the form $\mathrm{a}\theta(\mathrm{a})$ for $\mathrm{a} \in \mathrm{A}$.

Example

Let $\theta : \mathrm{a} \mapsto \mathrm{a},\ \mathrm{b} \mapsto \mathrm{d},\ \mathrm{c} \mapsto \mathrm{c},\ \mathrm{d} \mapsto \mathrm{b}$.

The θ-reduction of the word $\mathrm{d}\not{\mathrm{a}}\not{\mathrm{a}}\mathrm{acd}\!\!\!/\mathrm{b}\!\!\!/$ is dac.

Let $\theta : A \to A$ be an involution (possibly with some fixed point).

A word is *θ-reduced* if it has no factor of the form $a\theta(a)$ for $a \in A$.

Example

Let $\theta : a \mapsto a,\ b \mapsto d,\ c \mapsto c,\ d \mapsto b$.

The θ-reduction of the word $d\,aaacdb$ is dac.

A set is called *θ-symmetric* if it is closed under taking inverses (under θ).

Example

The set $X = \{a, adc, b, cba, d\}$ is symmetric for $\theta : b \leftrightarrow d$ fixing a, c.

Let $\theta : A \to A$ be an involution (possibly with some fixed point).

A word is *θ-reduced* if it has no factor of the form $a\theta(a)$ for $a \in A$.

Example

Let $\theta : a \mapsto a,\ b \mapsto d,\ c \mapsto c,\ d \mapsto b$.

The θ-reduction of the word $daaacdb$ is dac.

A set is called *θ-symmetric* if it is closed under taking inverses (under θ).

Example

The set $X = \{a, adc, b, cba, d\}$ is symmetric for $\theta : b \leftrightarrow d$ fixing a, c.

Let $\theta : A \to A$ be an involution (possibly with some fixed point).

A word is *θ-reduced* if it has no factor of the form $a\theta(a)$ for $a \in A$.

Example

Let $\theta : a \mapsto a,\ b \mapsto d,\ c \mapsto c,\ d \mapsto b$.

The θ-reduction of the word $d\,aaacdb$ is dac.

A set is called *θ-symmetric* if it is closed under taking inverses (under θ).

Example

The set $X = \{a, adc, b, cba, d\}$ is symmetric for $\theta : b \leftrightarrow d$ fixing a, c.

Let $\theta : A \to A$ be an involution (possibly with some fixed point).

A word is *θ-reduced* if it has no factor of the form $a\theta(a)$ for $a \in A$.

Example

Let $\theta : a \mapsto a, \; b \mapsto d, \; c \mapsto c, \; d \mapsto b$.

The θ-reduction of the word $d\,aaacdb$ is dac.

A set is called *θ-symmetric* if it is closed under taking inverses (under θ).

Example

The set $X = \{a, adc, b, cba, d\}$ is symmetric for $\theta : b \leftrightarrow d$ fixing a, c.

A *specular set* on an alphabet A (w.r.t. an involution θ) is a set

- biextendable,
- θ-symmetric,
- θ-reduced,
- tree set of characteristic 2.

A *specular set* on an alphabet A (w.r.t. an involution θ) is a set

- biextendable,
- θ-symmetric,
- θ-reduced,
- tree set of characteristic 2.

Example

Let $A = \{a, b\}$ and θ be the identity on A. The set of factors of $(ab)^\omega$ is a specular set.

A *specular set* on an alphabet A (w.r.t. an involution θ) is a set

- biextendable,
- θ-symmetric,
- θ-reduced,
- tree set of characteristic 2.

Example

Let $A = \{a, b\}$ and θ be the identity on A. The set of factors of $(ab)^\omega$ is a specular set.

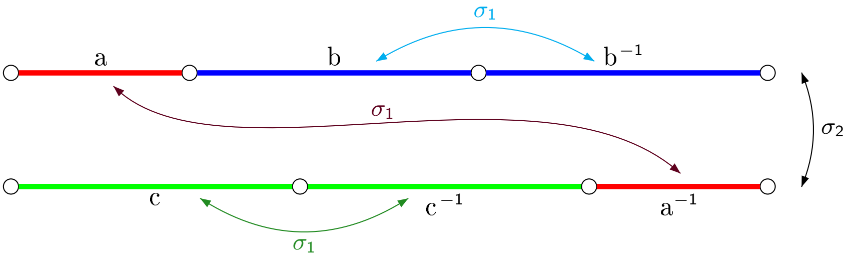
Proposition [*using* J. Cassaigne (1997)]

The factor complexity of a specular set is given by $p_n = n(\text{Card}(A) - 2) + 2$ for all $n \geq 1$.

Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

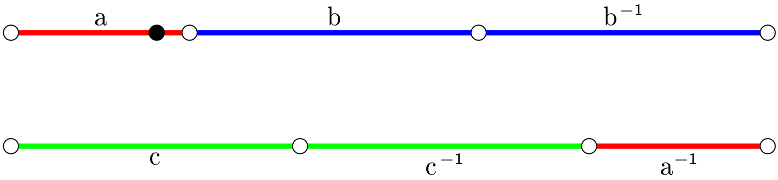
The natural coding of a linear involution without connections is a specular set.

$$T = \sigma_2 \circ \sigma_1$$



Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

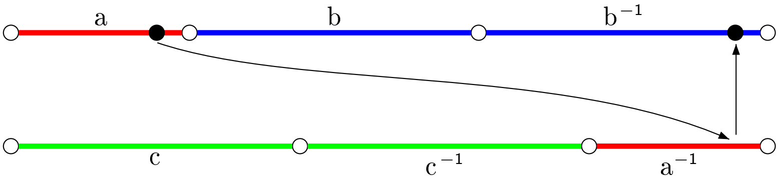
The natural coding of a linear involution without connections is a specular set.



$\Sigma T(z) = a$

Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

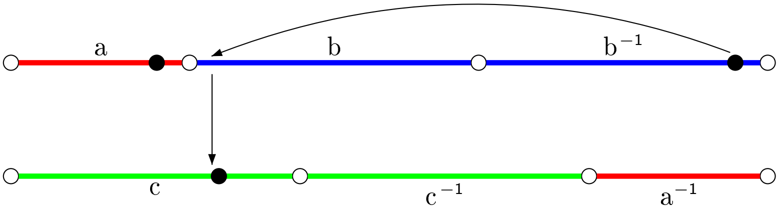
The natural coding of a linear involution without connections is a specular set.



$\Sigma T(z) = a\,b^{-1}$

Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

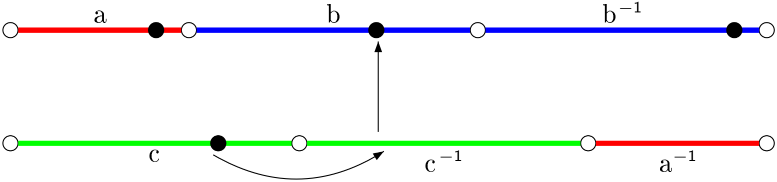
The natural coding of a linear involution without connections is a specular set.



$\Sigma T(z) = a\, b^{-1}\, c$

Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

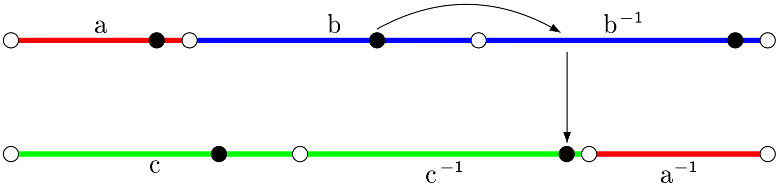
The natural coding of a linear involution without connections is a specular set.



$$\Sigma T(z) = a\, b^{-1}\, c\, b$$

Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

The natural coding of a linear involution without connections is a specular set.



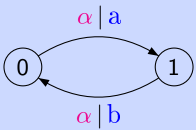
$$\Sigma T(z) = a\, b^{-1}\, c\, b\, c^{-1} \cdots$$

A *doubling transducer* is a transducer with set of states $\{0,1\}$ such that :

1. the input automaton is a group automaton,
2. the output labels of the edges are all distinct.

Example

$\Sigma = \{\alpha\}$
$A = \{a, b\}$



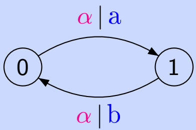
A *doubling transducer* is a transducer with set of states $\{0, 1\}$ such that :

1. the input automaton is a group automaton,
2. the output labels of the edges are all distinct.

A *doubling map* is a pair $\delta = (\delta_0, \delta_1)$, where $\delta_i(u) = v$ for a path starting at the state i with input label u and output label v.

Example

$\Sigma = \{\alpha\}$
$A = \{a, b\}$



$\delta_0(\alpha^\omega) = (ab)^\omega$
$\delta_1(\alpha^\omega) = (ba)^\omega$

A *doubling transducer* is a transducer with set of states $\{0, 1\}$ such that :

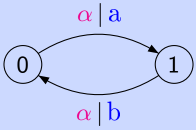
1. the input automaton is a group automaton,
2. the output labels of the edges are all distinct.

A *doubling map* is a pair $\delta = (\delta_0, \delta_1)$, where $\delta_i(u) = v$ for a path starting at the state i with input label u and output label v.

The *image* of a set T is $\delta(T) = \delta_0(T) \cup \delta_1(T)$.

Example

$\Sigma = \{\alpha\}$
$A = \{a, b\}$

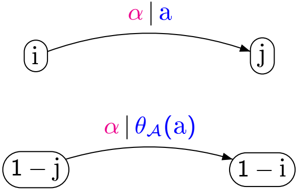


$\delta_0(\alpha^\omega) = (ab)^\omega$
$\delta_1(\alpha^\omega) = (ba)^\omega$

$\delta(\text{Fac}(\alpha^\omega)) = \text{Fac}((ab)^\omega)$

Proposition [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

The image of a tree set of characteristic 1 closed under reversal is a specular set with respect to $\theta_{\mathcal{A}}$.



Example (two doublings of Fibonacci on $\Sigma = \{\alpha, \beta\}$)

- Fac(abaababa$\cdots$) $\cup$ Fac(cdccdcdc$\cdots$)

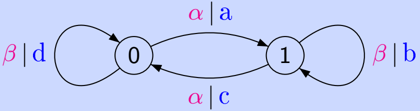
$$\theta_{\mathcal{A}} : \begin{cases} a \mapsto c \\ b \mapsto d \\ c \mapsto a \\ d \mapsto b \end{cases}$$

Example (two doublings of Fibonacci on $\Sigma = \{\alpha, \beta\}$)

- Fac (abaababa$\cdots$) $\cup$ Fac (cdccdcdc$\cdots$)

$$\theta_{\mathcal{A}} : \begin{cases} a \mapsto c \\ b \mapsto d \\ c \mapsto a \\ d \mapsto b \end{cases}$$

- Fac (abcabcda$\cdots$) $\cup$ Fac (cdacdabc$\cdots$)



$$\theta_{\mathcal{A}} : \begin{cases} a \mapsto a \\ b \mapsto d \\ c \mapsto c \\ d \mapsto b \end{cases}$$

A *right return word* to w in S is a nonempty word u such that $wu \in S$, starts and ends with w but has no w as an internal factor. Formally,

$$\mathcal{R}(w) = \{u \in A^+ \mid wu \in (A^+w \setminus A^+wA^+) \cap S\}.$$

Example (Fibonacci)

$\mathcal{R}(aa) = \{ b\underline{aa}, \ bab\underline{aa} \}$.

$$\varphi(a)^\omega = abaababa\underline{aa}baababaababaab\underline{aa}babaabaab\cdots$$

A *right return word* to w in S is a nonempty word u such that $wu \in S$, starts and ends with w but has no w as an internal factor. Formally,

$$\mathcal{R}(w) = \{u \in A^+ \mid wu \in (A^+w \setminus A^+wA^+) \cap S\}.$$

Example (Fibonacci)

$\mathcal{R}(aa) = \{ b\underline{aa},\ bab\underline{aa} \}$.

$$\varphi(a)^\omega = abaababa\underline{aa}baababaababaab\underline{aa}babaabaab\cdots$$

Cardinality Theorem for Right Return Words [BDDDLPRR (2015)]

For any w in a recurrent specular set, one has

$$\text{Card}\,(\mathcal{R}(w)) = \text{Card}\,(A) - 1.$$

A *complete return word* to a set $X \subset S$ is a word starting and ending with a word of X but having no internal factor in X. Formally,

$$\mathcal{CR}(X) = S \cap (XA^+ \cap A^+X) \setminus A^+XA^+.$$

Example (Fibonacci)

$\mathcal{CR}(\{aa, bab\}) = \{aabaa,\ aabab,\ babaa\}$.

$$\varphi(a)^\omega = abaababaabaababaababaabaababaabaab\cdots$$

A *complete return word* to a set $X \subset S$ is a word starting and ending with a word of X but having no internal factor in X. Formally,

$$\mathcal{CR}(X) = S \cap (XA^+ \cap A^+X) \setminus A^+XA^+.$$

Example (Fibonacci)

$\mathcal{CR}(\{aa, bab\}) = \{aabaa, aabab, babaa\}$.

$$\varphi(a)^\omega = abaababaabaababaababaabaababaabaab\cdots$$

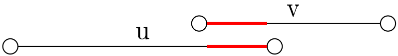
Cardinality Theorem for Complete Return Words [BDDDLPRR (2015)]

Let S be a recurrent specular set and $X \subset S$ be a finite *bifix code*[1] with empty *kernel*[2]. Then,

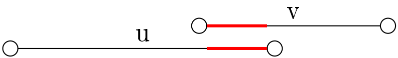
$$\text{Card}\,(\mathcal{CR}(X)) = \text{Card}\,(X) + \text{Card}\,(A) - 2.$$

1. *bifix code* : set that does not contain any proper prefix or suffix of its elements.
2. *kernel* : set of words of X which are also internal factors of X.

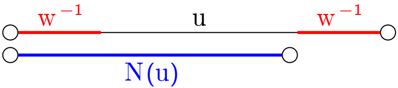
Two words u, v *overlap* if a nonempty suffix of one of them is a prefix of the other.



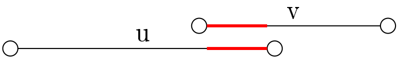
Two words u, v *overlap* if a nonempty suffix of one of them is a prefix of the other.



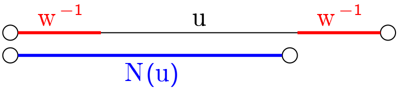
Consider a word w not overlapping with w^{-1}.
A *mixed return word* to w is the word $N(u)$ obtained from $u \in \mathcal{CR}(\{w, w^{-1}\})$ erasing the prefix if it is w and the suffix if it is w^{-1}.



Two words u, v *overlap* if a nonempty suffix of one of them is a prefix of the other.



Consider a word w not overlapping with w^{-1}.
A *mixed return word* to w is the word $N(u)$ obtained from $u \in \mathcal{CR}(\{w, w^{-1}\})$ erasing the prefix if it is w and the suffix if it is w^{-1}.



Cardinality Theorem for Mixed Return Words [BDDDLPRR (2015)]

Let S be a recurrent specular set and $w \in S$ such that w, w^{-1} do not overlap. Then,

$$\text{Card}\,(\mathcal{MR}(w)) = \text{Card}\,(A).$$

A *palindrome* is a word $w = \widetilde{w}$ as, for instance :

A *palindrome* is a word $w = \widetilde{w}$ as, for instance :

 eye, noon, sagas, racecar . . .

A *palindrome* is a word $w = \widetilde{w}$ as, for instance :

eye, noon, sagas, racecar . . .

ici, été, coloc, kayak, radar, . . .

A *palindrome* is a word $w = \widetilde{w}$ as, for instance :

eye, noon, sagas, racecar ...

ici, été, coloc, kayak, radar, ...

non, osso, aveva, rossor, ottetto, ...

A *palindrome* is a word $w = \widetilde{w}$ as, for instance :

 eye, noon, sagas, racecar ...

 ici, été, coloc, kayak, radar, ...

 non, osso, aveva, rossor, ottetto, ...

 Jelenovi Pivo Nelej (*to a deer, don't pour beer*),

A *palindrome* is a word $w = \widetilde{w}$ as, for instance :


eye, noon, sagas, racecar ...


ici, été, coloc, kayak, radar, ...


non, osso, aveva, rossor, ottetto, ...


Jelenovi Pivo Nelej (*to a deer, don't pour beer*), Ital Platí, ...

Theorem [X. Droubay, J. Justin, G. Pirillo (2001)]

A word of length n has at most $n+1$ palindrome factors.

A word with maximal number of palindromes is *rich*.
A factorial set is *rich* if all its elements are rich.

Example (Fibonacci)

$\mathsf{Pal}(abaab) = \{\varepsilon, a, b, aa, aba, baab\}$.

Theorem [X. Droubay, J. Justin, G. Pirillo (2001)]

A word of length n has at most $n+1$ palindrome factors.

A word with maximal number of palindromes is *rich*.
A factorial set is *rich* if all its elements are rich.

Example (Fibonacci)

$\mathsf{Pal}(abaab) = \{\varepsilon, a, b, aa, aba, baab\}$.

Theorem [A. Glen, J. Justin, S. Widmer, L.Q. Zamboni (2009)]

Let S be a recurrent set closed under reversal.
S is rich $\iff$ every complete return word to a palindrome is a palindrome.

Theorem

Families of rich sets :

- Factors of Arnoux-Rauzy (*Sturmian*) words.

  [X. Droubay, J. Justin, G. Pirillo (2001)]

- Natural coding of regular interval exchanges defined by a symmetric permutation.

  [P. Balázi, Z. Masáková, E. Pelantová (2007)]

Theorem

Families of rich sets :

- Factors of Arnoux-Rauzy (*Sturmian*) words.

  [X. Droubay, J. Justin, G. Pirillo (2001)]

- Natural coding of regular interval exchanges defined by a symmetric permutation.

  [P. Balázi, Z. Masáková, E. Pelantová (2007)]

Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2016)]

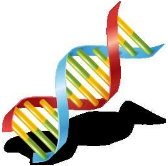
Recurrent tree sets of characteristic 1 closed under reversal are rich.

Let σ be an antimorphism.
A word w is a *σ-palindrome* if $\mathrm{w} = \sigma(\mathrm{w})$.

Let σ be an antimorphism.
A word w is a *σ-palindrome* if $w = \sigma(w)$.

Example

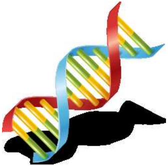
Let σ : $\mathtt{A} \leftrightarrow \mathtt{T}$, $\mathtt{C} \leftrightarrow \mathtt{G}$.
The word $\mathtt{CTTAAG}$ is a σ-palindrome.



Let σ be an antimorphism.
A word w is a *σ-palindrome* if $w = \sigma(w)$.

Example

Let σ : A $\leftrightarrow$ T, C $\leftrightarrow$ G.
The word CTTAAG is a σ-palindrome.



Theorem [Š. Starosta (2011)]

Let $\gamma_\sigma(w)$ be the number of transpositions of σ affecting w. Then,

$$\text{Card}\,(\text{Pal}_\sigma(w)) \leq |w| + 1 - \gamma_\sigma(w).$$

A word (set) is *σ-rich* if the equality holds (for all its elements).

Let G be a group of morphisms and antimorphisms, containing at least an antimorphism.
A word w is a G-*palindrome* if there exists a nontrivial $g \in G$ s.t. $w = g(w)$.

Let G be a group of morphisms and antimorphisms, containing at least an antimorphism. A word w is a G-*palindrome* if there exists a nontrivial $g \in G$ s.t. $w = g(w)$.

Example

Let $G = \langle \sigma, \tau \rangle$, with $\sigma : \mathtt{A} \leftrightarrow \mathtt{R},\ \mathtt{E} \leftrightarrow \mathtt{T},\ \mathtt{I} \leftrightarrow \mathtt{M},\ \mathtt{O} \leftrightarrow \mathtt{U}$ and $\tau : \mathtt{A} \leftrightarrow \mathtt{G},\ \mathtt{E} \leftrightarrow \mathtt{P},\ \mathtt{U} \leftrightarrow \mathtt{R}$.

The following words are G-palindromes :

Let G be a group of morphisms and antimorphisms, containing at least an antimorphism. A word w is a G-*palindrome* if there exists a nontrivial $g \in G$ s.t. $w = g(w)$.

Example

Let $G = \langle \sigma, \tau \rangle$, with $\sigma : \mathtt{A} \leftrightarrow \mathtt{R},\ \mathtt{E} \leftrightarrow \mathtt{T},\ \mathtt{I} \leftrightarrow \mathtt{M},\ \mathtt{O} \leftrightarrow \mathtt{U}$ and $\tau : \mathtt{A} \leftrightarrow \mathtt{G},\ \mathtt{E} \leftrightarrow \mathtt{P},\ \mathtt{U} \leftrightarrow \mathtt{R}$.

The following words are G-palindromes :

- NUMERATION, fixed by σ,

Let G be a group of morphisms and antimorphisms, containing at least an antimorphism. A word w is a G*-palindrome* if there exists a nontrivial $g \in G$ s.t. $w = g(w)$.

Example

Let $G = \langle \sigma, \tau \rangle$, with $\sigma : \mathtt{A} \leftrightarrow \mathtt{R},\ \mathtt{E} \leftrightarrow \mathtt{T},\ \mathtt{I} \leftrightarrow \mathtt{M},\ \mathtt{O} \leftrightarrow \mathtt{U}$ and
$\tau : \mathtt{A} \leftrightarrow \mathtt{G},\ \mathtt{E} \leftrightarrow \mathtt{P},\ \mathtt{U} \leftrightarrow \mathtt{R}$.

The following words are G-palindromes :

- NUMERATION, fixed by σ,
- PRAGUE, fixed by τ,

Let G be a group of morphisms and antimorphisms, containing at least an antimorphism. A word w is a G*-palindrome* if there exists a nontrivial $\mathrm{g} \in \mathrm{G}$ s.t. $\mathrm{w} = \mathrm{g}(\mathrm{w})$.

Example

Let $\mathrm{G} = \langle \sigma, \tau \rangle$, with $\sigma : \mathtt{A} \leftrightarrow \mathtt{R},\ \mathtt{E} \leftrightarrow \mathtt{T},\ \mathtt{I} \leftrightarrow \mathtt{M},\ \mathtt{O} \leftrightarrow \mathtt{U}$ and $\tau : \mathtt{A} \leftrightarrow \mathtt{G},\ \mathtt{E} \leftrightarrow \mathtt{P},\ \mathtt{U} \leftrightarrow \mathtt{R}$.

The following words are G-palindromes :

- NUMERATION, fixed by σ,
- PRAGUE, fixed by τ,
- PÍT, fixed by $\sigma\tau\sigma$.

Let G be a group of morphisms and antimorphisms, containing at least an antimorphism. A word w is a G*-palindrome* if there exists a nontrivial $g \in G$ s.t. $w = g(w)$.

Example

Let $G = \langle \sigma, \tau \rangle$, with $\sigma : \mathtt{A} \leftrightarrow \mathtt{R},\ \mathtt{E} \leftrightarrow \mathtt{T},\ \mathtt{I} \leftrightarrow \mathtt{M},\ \mathtt{O} \leftrightarrow \mathtt{U}$ and $\tau : \mathtt{A} \leftrightarrow \mathtt{G},\ \mathtt{E} \leftrightarrow \mathtt{P},\ \mathtt{U} \leftrightarrow \mathtt{R}$.

The following words are G-palindromes :

- NUMERATION, fixed by σ,
- PRAGUE, fixed by τ,
- PÍT, fixed by $\sigma\tau\sigma$.

A word (set) is G*-rich* * if...

Let G be a group of morphisms and antimorphisms, containing at least an antimorphism. A word w is a G*-palindrome* if there exists a nontrivial $\mathrm{g} \in \mathrm{G}$ s.t. $\mathrm{w} = \mathrm{g}(\mathrm{w})$.

Example

Let $\mathrm{G} = \langle \sigma, \tau \rangle$, with $\sigma : \mathtt{A} \leftrightarrow \mathtt{R},\ \mathtt{E} \leftrightarrow \mathtt{T},\ \mathtt{I} \leftrightarrow \mathtt{M},\ \mathtt{O} \leftrightarrow \mathtt{U}$ and
$\tau : \mathtt{A} \leftrightarrow \mathtt{G},\ \mathtt{E} \leftrightarrow \mathtt{P},\ \mathtt{U} \leftrightarrow \mathtt{R}$.

The following words are G-palindromes :

- NUMERATION, fixed by σ,
- PRAGUE, fixed by τ,
- PÍT, fixed by $\sigma\tau\sigma$.

A word (set) is G*-rich* * if... "the number of G-palindromes if maximal".

Theorem [E. Pelantová, Š. Starosta (2014)]

A set S closed under G is G-rich if for every $w \in S$, every complete return word to the G-orbit of w is fixed by a nontrivial element of G.

Theorem [E. Pelantová, Š. Starosta (2014)]

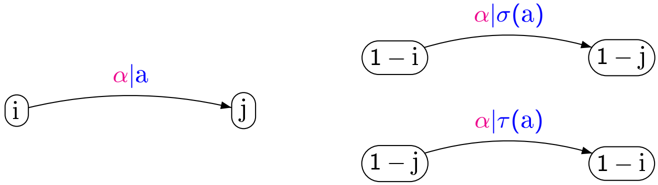
A set S closed under G is G-rich if for every $w \in S$, every complete return word to the G-orbit of w is fixed by a nontrivial element of G.

Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2016)]

The specular set obtained as image under a doubling transducer $\mathcal{A}$ is $G_{\mathcal{A}}$-rich.

$$G_{\mathcal{A}} = \{\mathrm{id}, \sigma, \tau, \sigma\tau\} \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$$

with σ an antimorphism and τ a morphism.



Conclusions

Summing up

- Tree and specular sets.

  Linear involutions and doubling maps.

Conclusions

Summing up

- Tree and specular sets.

  Linear involutions and doubling maps.

- Cardinality Theorems for return words.

$$\begin{aligned}\text{Card}\,(\mathcal{R}(\mathrm{w})) &= \text{Card}\,(\mathrm{A}) - 1\\ \text{Card}\,(\mathcal{CR}(\mathrm{X})) &= \text{Card}\,(\mathrm{X}) + \text{Card}\,(\mathrm{A}) - 2\\ \text{Card}\,(\mathcal{MR}(\mathrm{w})) &= \text{Card}\,(\mathrm{A})\end{aligned}$$

Conclusions

Summing up

- Tree and specular sets.

  Linear involutions and doubling maps.

- Cardinality Theorems for return words.

$$\begin{aligned} \text{Card}\,(\mathcal{R}(w)) &= \text{Card}\,(A) - 1 \\ \text{Card}\,(\mathcal{CR}(X)) &= \text{Card}\,(X) + \text{Card}\,(A) - 2 \\ \text{Card}\,(\mathcal{MR}(w)) &= \text{Card}\,(A) \end{aligned}$$

- New family of G-rich sets.

  Specular sets obtained by doubling maps are $G_{\mathcal{A}}$-rich.

Further Research Directions

and other works in progress

- Decidability of the tree (and specular) condition.

  [work in progress with Julien Leroy and Revekka Kyriakoglou]

Further Research Directions

and other works in progress

- Decidability of the tree (and specular) condition.

  [work in progress with Julien Leroy and Revekka Kyriakoglou]

- Tree set and free groups.

  Tree set of $\chi = 1 \Longrightarrow \mathcal{R}(\mathrm{w})$ is a basis of the free group for every w

Further Research Directions

and other works in progress

- Decidability of the tree (and specular) condition.

  [work in progress with Julien Leroy and Revekka Kyriakoglou]

- Tree set and free groups.

  Tree set of $\chi = 1 \implies \mathcal{R}(\mathrm{w})$ is a basis of the free group for every w

- New classes of G-rich sets (or new groups G).

Děkuji