Recognizability

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Motivation

Let x be the Fibonacci word, defined as the fixed point $f^{\omega}(a)$ of the substitution $f : a \mapsto ab$, $b \mapsto a$, that is

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Can we decompose x as a concatenantion of $f(a) = ab$ and $f(b) = a$?

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More in general, given a word $v \in \mathcal{L}(x)$, can we decompose v as a concatenation (up to a prefix and a suffix) of ab and a ? Is this decomposition unique ?

$$v = ab|ab|a|ab|a_{b}^{a}$$

Natural cutting points

Let σ be a primitive substitution over an alphabet A. Let $u = u_0 u_1 \cdots$ be a fixed point of σ. For each $k > 0$ let define

$$E_k = \{0\} \cup \{|\sigma^k(u_0 \cdots u_{p-1})| : p > 0\},$$

the set of *natural k-cutting points* (or *cutting bars* of order k).

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Example

Let x be the Fibonacci word.

$$x = \quad ab\ a\ \ ab\ \ ab\ a\ \ ab\ \ a\ \quad ab\ \quad ab\ \ a\ \quad ab\ \quad ab\ \ a \cdots$$

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- $E_1 = \{0, 2, 3, 5, 7, 8, 10, 11, 13, 15, 16, 18, 20, \ldots\}$

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- $E_1 = \{0, 2, 3, 5, 7, 8, 10, 11, 13, 15, 16, 18, 20, \ldots\}$
- $E_2 = \{0, 3, 5, 8, 11, 13, 16, 18, 21, \ldots\}$

Natural cutting points

Some easy remarks

$$E_k = \{0\} \cup \{|\sigma^k(u_0 \cdots u_{p-1})| : p > 0\}$$

- If $k \leq \ell$, then $E_\ell \subset E_k$.
- If σ has constant length q, then $E_k = \{mq^k \ : \ m \geq 0\}$.

Example

Let u be the Morse word defined as the fixed point $\mu^\omega(a)$ of the substitution $\mu : a \mapsto ab$, $b \mapsto ba$. Then $E_1 = \{0, 2, 4, 6, \ldots\}$.

$$u = |ab|ba|ba|ab|ba|ab|ab|ba|ba|ab|ab|ba|ab| \cdots$$

Cutting points and ancestors

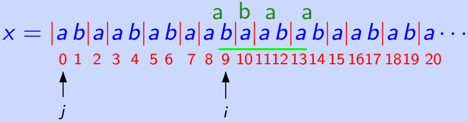
For every factor $v = u_i \cdots i_{i+|v|-1}$, there exists a *rank* j, a *length* ℓ, a suffix S of $\sigma(u_j)$ and a prefix P of $\sigma(u_{j+\ell+1})$ such that $v = S\,\sigma(u_{j+1}) \cdots \sigma(u_{j+\ell})\,P$, and such that

$$E_k \cap \{i, \dots, i + |v| - 1\} \;=\; (i - h) + E_k \cap \{h, \dots, h + |v| - 1\},$$

with $h = |\sigma(u_0 \cdots u_j)| - |S|$.
We say that $[S, \sigma^k(u_{j+1}), \dots, \sigma^k(u_{j+\ell}), P]$ is the *k-cutting at the rank i* of v, and that v comes from the *ancester word* $u_j \cdots u_{j+\ell+1}$.

Example

Let $v = baaba$ a factor of the Fibonacci word.



The 1-cutting at rank 9 of v is $[b, f(b), f(a), a]$ and his ancestor is $abaa$.

Unilaterally recognizable substitutions

A substitution σ is called *unilaterally* (right) *recognizable* if there exists an integer $L > 0$ such that

$$\begin{cases} u_i u_{i+1} \cdots u_{i+L-1} = u_j u_{j+1} \cdots u_{j+L-1} \\ i \in E_1 \end{cases} \qquad \Longrightarrow \qquad j \in E_1.$$

The smaller inter L that verifies this property is called the *recognizability index* of σ.

Then, a substitution is recognizable if the 1-cutting of any long enough factor is independent of the rank of occurrence of the factor, except maybe for a suffix of the word.

Unilaterally recognizable substitutions

Some examples

Examples

- The substitution $\sigma : a \mapsto aba,\ b \mapsto bab$ is not recognizable (and so is any infinite periodic word).

$$\sigma^{\omega}(a) = |\overline{aba|b}aba|b\overline{ab|ab}a|bab|aba|bab|aba|ba\cdots$$

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$$\sigma^{\omega}(a) = |\overline{aba|b}aba|b\overline{ab|ab}a|bab|aba|bab|aba|ba\cdots$$

- The Fibonacci substitution is recognizable with recognizability index 2.

$$x = |ab|a|ab|ab|a|ab|a|ab|ab|a|ab|ab|a\cdots$$

Unilaterally recognizable substitutions

The Morse substitution

Example [M. Queffélec, 1987]

The Morse substitution is recognizabile.

$$u = |ab|ba|ba|ab|ba|ab|ab|ba|ba|ab|ab|ba|ab| \cdots$$

Indeed, the question is to decide wether some a (resp. b) is the beginning of some substituted word, $\mu(a)$ (resp. $\mu(b)$) necessarily.

- We can say nothing for the letter a.
- aa and is not a substituted word, while ab occurs both as a factor of $\mu(a)$ and $\mu(ba) = baba$.
- No word with more than two consecutive b occurs in u; it follows that abb is the beginning of some substituted word but aba may not be one, since it appears both in $\mu(aa) = abab$ and $\mu(bb) = baba$.
- $abaa$ is never a substituted word (since aa is not one), while $abab$ is always $\mu(aa)$.

We proved that a is the beginning of $\mu(a)$ in u, if it appears in abb or $abab$. By symmetryn we conclude that the (unilaterally) recognizability index is 4.

A brief history . . .

- In 1973, J.C. Martin claims that any substitution on a two-letter alphabet which is aperiodic is unilaterally recognizable (or *rank one determined*). His proof is not convincing.
- In 1987, M. Queffélec announces a short proof of the unilaterally recognizability of constant length substitutions due to G. Rauzy. Nobody could check this proof.
- In his 1989 PhD Thesis, M. Mentzen said to prove this result, using a paper by T. Kamae of 1972.
- In 1999, C. Apparicio shows a gap in Mentzen proof (Kamae's results only works for a particular case of the theorem, namely if the length is a power of a prime number). She solves the problem using a 1978 result by F.M. Dekking.
- In the meantime, in 1992, B. Mossé proves a more general result (also nonconstant length), but using a new notion of (bilaterally) recognizable substitution (see later). She refines this result in 1996.

Recognizability of constant length substitutions

Theorem [Mentzen, 1989 – Apparicio 1999]

Let σ a constant length substitution, one-to-one over the alphabet, with fixed point $u = \sigma^{\omega}(a)$ aperiodic, and satisfying

$$\forall\, b \in A, \exists\, k \geq 1 \text{ such that } a \text{ occurs in } \sigma^{k}(b). \tag{1}$$

Then σ is unilaterally recognizable.

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Remarks.

- If σ is primitive then condition (1) is verified.

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Remarks.

- If σ is primitive then condition (1) is verified.
- A q-constant length substitution σ is unilaterally recognizabile if there exists an $L > 0$ such that for every $n, i, t \in \mathbb{N}$,

$$u_i \cdots u_{i+Lq^n-1} = u_{tq^n} \cdots u_{(t+L)q^n-1} \quad \Longrightarrow \quad q^{n} \text{ divides } i$$

A^{ω} is a compact metric space, with distance $d(x, y) = \frac{1}{\min\{m+1 \mid x_m \neq y_m\}}$.
The set $W = \overline{\mathcal{O}_T(u)}$ is a compact. Moreover, if σ verifies condition (1), W is a minimal set of (A^{ω}, T).

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Lemma 1

For every point of W one has $\sigma^n \circ T^p = T^{pq^n} \circ \sigma^p$, for every $n, p \in \mathbb{N}$.

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Proposition 2 [Dekking, 1978]

For every $n \in \mathbb{N}$, $W = \cup_{j=0}^{q^n-1} T^j \circ \sigma^n(W)$, where the union is disjoint.

1^{st} step. Let first show that for every $n \in \mathbb{N}$, there exists an L_n such that for every $t, i \in \mathbb{N}$,

$$u_i \cdots u_{i+L_n q^n - 1} = u_{tq^n} \cdots u_{tq^n + L_n q^n - 1} \quad \Longrightarrow \quad q^n \text{ divides } i.$$

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Let suppose, by contraddiction, that there exists two sequences $(r(t))_{t\in\mathbb{N}}$ and $(s(t))_{t\in\mathbb{N}}$ such that $r(t) \equiv 0 \pmod{q^n}$ and $s(t) \not\equiv 0 \pmod{q^n}$, verifying

$$u_{s(t)} \cdots u_{s(t)+tq^n-1} = u_{r(t)} \cdots u_{r(t)+tq^n-1}$$

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Since W is compact, there exist two convergent subsequences $T^{r(t_i)}u \to x$ and $T^{s(t_i)}u \to y$. Since $r(t_i) \equiv 0 \pmod{q^n}$ and $\sigma^n(W)$ is closed then $x \in \sigma^n(W)$ (by Lemma 1). Similary, $y \in T^p \circ \sigma^n(W)$ for a $p \not\equiv 0 \pmod{q^n}$. Thus $x \neq y$ by Theorem 1.

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But $d(T^{r(t)}u, T^{s(t)}u) \leq \frac{1}{tq^n} \to 0$ when $t \to \infty$. Thus $x = y$, a contradiction.

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In particular, if we consider the last $L_1q - 1$ letters, we have

$$u_{i+L_1q(q-1)} \cdots u_{i+L_1q(q-1)+L_1q^1-1} = u_{(tq+L_1(q-1))q^1} \cdots u_{(tq+L_1(q-1))q^1+L_1q^1-1},$$

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thus, q divides $i + L_1q(q-1)$, whence q divides i.

Let $i = qs$. We show that q divides s. Since $\sigma(u) = u$, one has

$$u_i \cdots u_{i+L_1q^2-1} = \sigma(u_s \cdots u_{s+L_1q-1}),\ u_{tq^2} \cdots u_{tq^2+L_1q^2-1} = \sigma(u_{tq} \cdots u_{tq+L_1q-1}).$$

By injectivity and by definition of L_1 one obtain that q divides s.

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Using the same reasoning, one has $L_1 = L_2 = \ldots = L_n = \ldots$.

Unilaterally recognizability

Theorem [B. Host, 1986]

Let σ a primitive substitution and X_σ the associated dynamical system. The substitution σ is (unilaterally) recognizable if and only if $\sigma(X_\sigma)$ is an open set.

Substitutions not unilaterally recognizable

A sufficient condition for a (non periodic) substitution not to be unilaterally recognizable is that for every couple of distinct letter (a, b), $\sigma(a)$ is a strict suffix of $\sigma(b)$, or conversely.

Example

The substitution $\sigma : a \mapsto aaab, \quad b \mapsto ab$ is not unilaterally recognizable.

Example

The *Chacon substitution* over the alphabet $\{0, 1, 2\}$ defined by

$$0 \mapsto 0012, \quad 1 \mapsto 12, \quad 2 \mapsto 012$$

is not unilaterally recognizable.

A negative result . . .

Theorem [B. Mossé, 1992]

Let σ be a primitive substitution with a non-periodic fixed point u. The substitution σ is <u>not</u> unilaterally recognizable if and only if for every $L > 0$ there exists a word v of length L and two letters a and b of the alphabet such that :

1. the word $\sigma(b)$ is a proper suffix of $\sigma(a)$;
2. the words $\sigma(a)v$ and $\sigma(b)v$ are both in $\mathcal{L}(u)$ and with the same 1-cutting of v.

Bilaterally recognizable substitutions

A substitution σ is called (*bilaterally*) *recognizable* if there exists an integer $L > 0$ such that

$$\left\{ \begin{array}{l} u_{i-L} \cdots u_{i+L} = u_{j-L} \cdots u_{j+L} \\ i \in E_1 \end{array} \right. \qquad \Longrightarrow \qquad j \in E_1.$$

The smaller inter L that verifies this property is called the *recognizability index* of σ.

Then, a substitution is bilaterally recognizable if the 1-cutting of any long enough factor is independent of the rank of occurrence of the factor, except maybe for a suffix and a prefix of the word.

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Theorem [B. Mossé, 1996]

Let σ be a primitive substitution with a non-periodic fixed point u. There exists an integer $L > 0$ such that if

$$u_{i-L} \cdots i_{j+L} = u_{i'-L} \cdots u_{j'+L},$$

then, $u_i \cdots u_j$ and $u_{i'} \cdots u_{j'}$ have the same 1-cutting at ranks i and i' and the same ancestor.

Some remarks on the periodicity

There exists nontrivial substitutions which are periodic.

Example

The fixed point $\sigma(a)$ of the substitution $\sigma : a \mapsto aba\ ,\ b \mapsto babab$ is the periodic point $(ab)^\omega$.

However the problem is decidable.

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The fixed point $\sigma(a)$ of the substitution $\sigma : a \mapsto aba$, $b \mapsto babab$ is the periodic point $(ab)^\omega$.

However the problem is decidable.

A primitive non-periodic substitution σ does not have necessarily a fixed point, but has at least a periodic point, that is a u such that $\sigma^k(u) = u$ for some $k > 0$. Thus, a power of σ is bilaterally recognizable.

Recognizability for biinfinite words

We can extend the notion to a two-sided version, defining

$$E_1 = \{0\} \cup \{|\sigma(u_0 \cdots u_{p-1})| : p > 0\} \cup \{|\sigma(u_{-p} \cdots u_{-1})| : p > 0\}$$

From Mossé's theorem, we can deduce that any word of the bilateral dynamical system X_σ associated with a primitive non-periodic substitution σ can be *desubstituted* in a unique biinfinite word.

Corollary

Let σ be a primitive non-periodic substitution. Let X_σ be the associated substitutive dynamical system. Then, for any $w \in X_\sigma$ there exists a unique $v \in X_\sigma$ such that $w = T^k\sigma(v)$, with $0 \leq k \leq |\sigma(v_0)|$.

$$w = \cdots \mid \underbrace{\cdots}_{\sigma(v_{-1})} \mid \underbrace{w_{-k} \cdots w_1 . w_0 \cdots w_\ell}_{\sigma(v_0)} \mid \underbrace{\cdots}_{\sigma(v_1)} \mid \underbrace{\cdots}_{\sigma(v_2)} \mid \cdots$$

where $v = \cdots v_{-1}.v_0 v_1 \cdots \in X_\sigma$.

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