Dendric languages and return words

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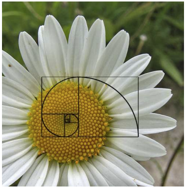
Konference *TIGR CoW 2021*

Telč, 27. června 2021

Fibonacci

$$\mathbf{x} = \text{abaababaabaababa}\cdots$$

$$\mathbf{x} = \lim_{n\to\infty} \varphi^n(a) \qquad \text{where} \qquad \varphi : \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$



Fibonacci

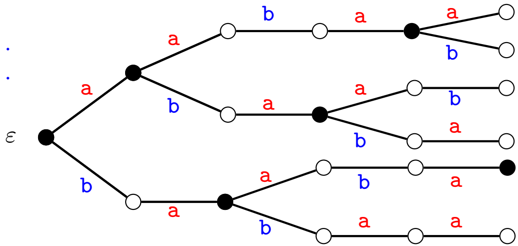
$$\mathbf{x} = \texttt{abaababaabaababa}\cdots$$

The *Fibonacci language* (set of factors of $\mathbf{x}$) is a Sturmian language.

Definition

A *Sturmian* language $\mathcal{L} \subset \mathcal{A}^*$ is a factorial set such that $p_n = \mathrm{Card}\,(\mathcal{L} \cap \mathcal{A}^n) = n + 1$.

n :	0	1	2	3	4	5	$\cdots$
p_n :	1	2	3	4	5	6	$\cdots$



2-coded Fibonacci

$\mathsf{x} = \texttt{ab aa ba ba ab aa ba ba} \ \cdots$

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$$f : \begin{cases} \texttt{u} & \mapsto & \texttt{aa} \\ \texttt{v} & \mapsto & \texttt{ab} \\ \texttt{w} & \mapsto & \texttt{ba} \end{cases}$$

2-coded Fibonacci

$$\mathsf{x} = \mathtt{ab}\ \mathtt{aa}\ \mathtt{ba}\ \mathtt{ba}\ \mathtt{ab}\ \mathtt{aa}\ \mathtt{ba}\ \mathtt{ba}\ \cdots$$

$$f^{-1}(\mathsf{x}) = \mathtt{v}\ \mathtt{u}\ \mathtt{w}\ \mathtt{w}\ \mathtt{v}\ \mathtt{u}\ \mathtt{w}\ \mathtt{w}\ \cdots$$

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Arnoux-Rauzy languages

Definition

An *Arnoux-Rauzy* language is a factorial set closed by reversal with $p_n = (\mathrm{Card}\,(\mathcal{A}) - 1)n + 1$ having a unique right special factor for each length.

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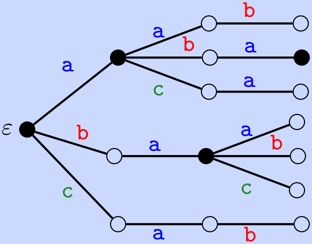
Example (Tribonacci)

Factors of the fixed point $\psi^\omega(\mathtt{a})$ of the morphism $\psi : \mathtt{a} \mapsto \mathtt{ab}, \quad \mathtt{b} \mapsto \mathtt{bc}, \quad \mathtt{c} \mapsto \mathtt{a}.$



$$\begin{array}{lcccccc} n: & 0 & 1 & 2 & 3 & \cdots \\ p_n: & 1 & 3 & 5 & 7 & \cdots \end{array}$$

$$p_n = 2n + 1$$

2-coded Fibonacci

$$f^{-1}(\mathsf{x}) = \texttt{v u w w v u w w} \cdots$$

Is the set of factors of $f^{-1}(\mathsf{x})$ an Arnoux-Rauzy language?

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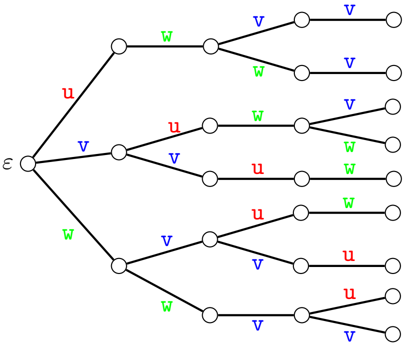


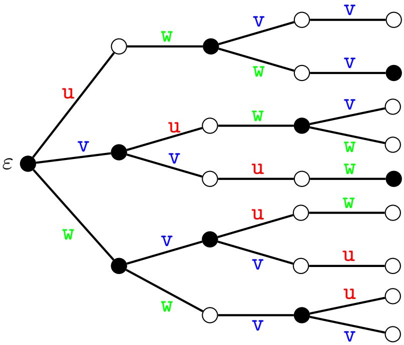
$$p_n = 2n + 1$$

n :	0	1	2	3	4	⋯
p_n :	1	3	5	7	9	⋯

2-coded Fibonacci

$$f^{-1}(\mathsf{x}) = \texttt{v u w w v u w w} \cdots$$

Is the set of factors of $f^{-1}(\mathsf{x})$ an Arnoux-Rauzy language? No!



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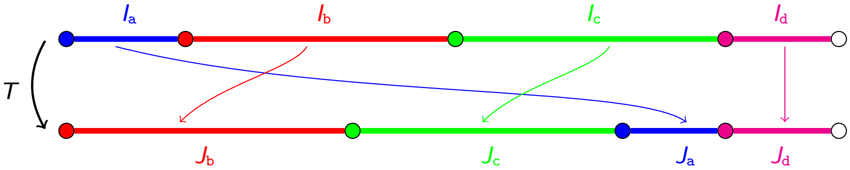
Interval exchanges

Let $(I_\alpha)_{\alpha\in\mathcal{A}}$ and $(J_\alpha)_{\alpha\in\mathcal{A}}$ be two partitions of $[0,1[$.
An *interval exchange transformation* (IET) is a map $T : [0,1[\to [0,1[$ defined by

$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha.$$



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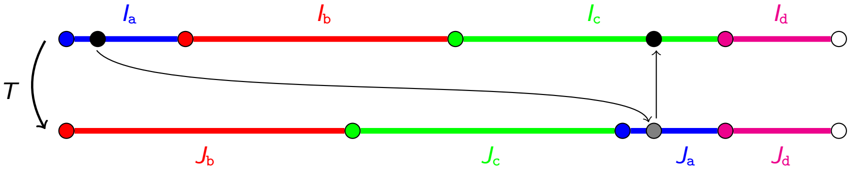

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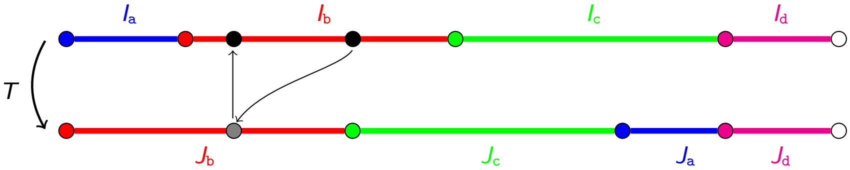

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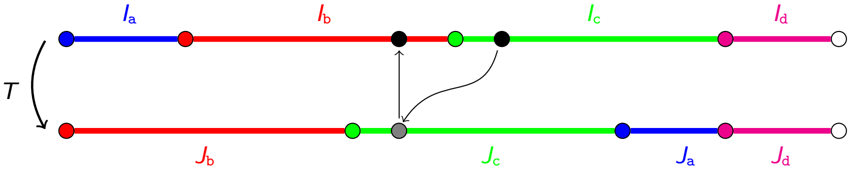

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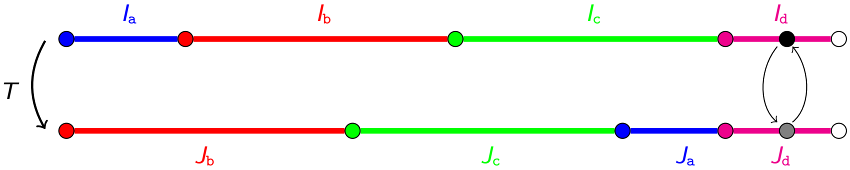

Interval exchanges

T is said to be *minimal* if for any point $z \in [0, 1[$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[0, 1[$.

T is said *regular* if the orbits of the non-zero separation points are infinite and disjoint.

Theorem [Keane (1975)]

A regular interval exchange transformation is minimal.

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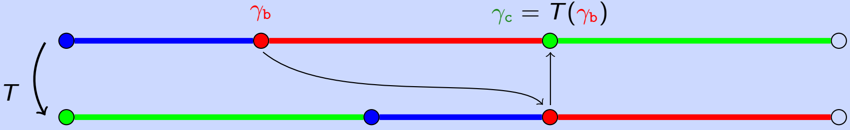
Example (the converse is not true)

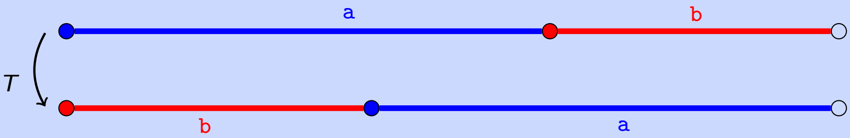


Interval exchanges

The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in \mathcal{A}^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha .$$

Example (Fibonacci, $z = (3-\sqrt{5})/2$)



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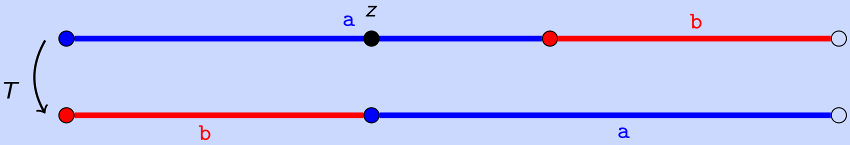

$\Sigma_T(z) = \mathtt{a}$

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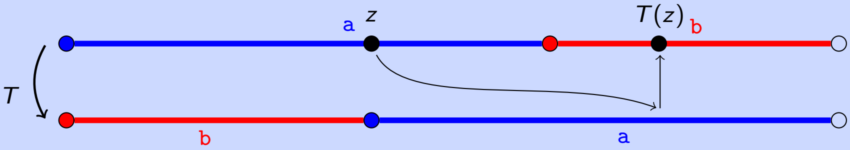

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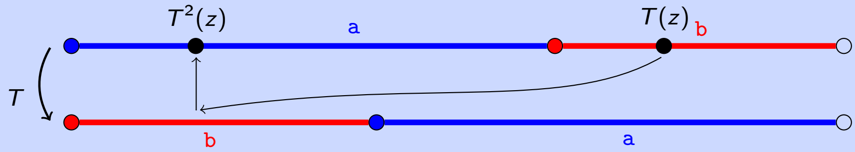

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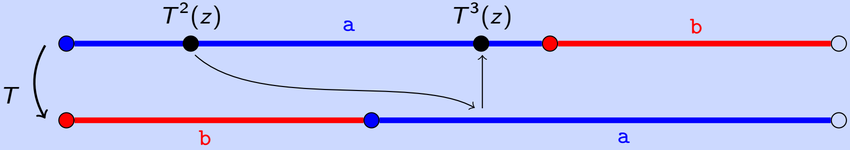

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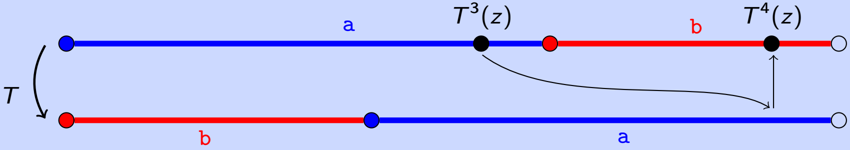

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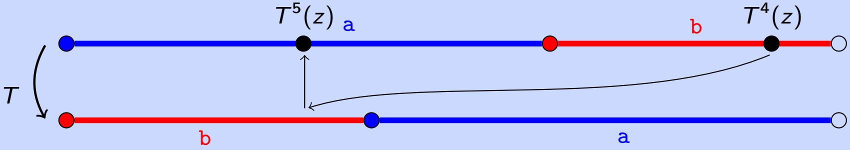

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Interval exchanges

The language $\mathcal{L}(T) = \bigcup_{z \in [0,1[} \mathrm{Fac}\,(\Sigma_T(z))$ is said a (*minimal*, *regular*) *interval exchange language*.

Remark. If T is minimal, $\mathrm{Fac}\,(\Sigma_T(z))$ does not depend on the point z.

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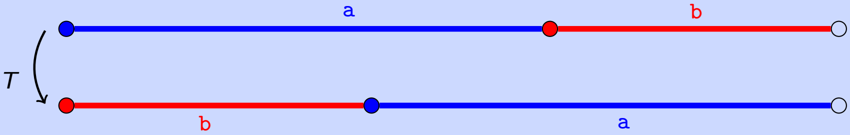

$$\mathcal{L}(T) = \left\{\varepsilon,\ \mathtt{a}, \mathtt{b},\ \mathtt{aa}, \mathtt{ab}, \mathtt{ba},\ \mathtt{aab}, \mathtt{aba}, \mathtt{baa}, \mathtt{bab}\ \ldots\right\}$$

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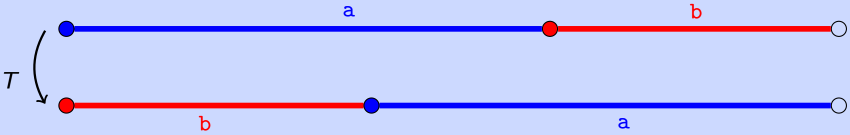

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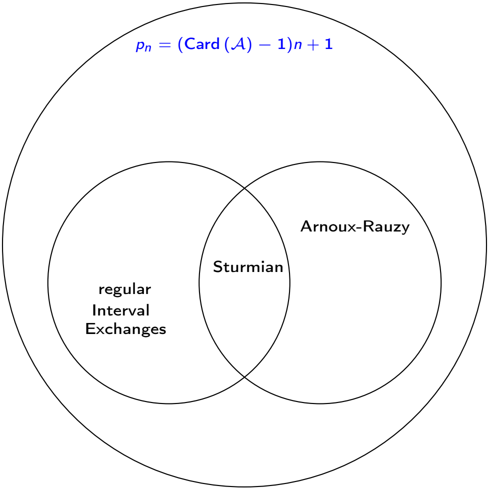
Proposition

Regular interval exchange languages have factor complexity $p_n = (\mathrm{Card}\,(\mathcal{A}) - 1)n + 1$.

Arnoux-Rauzy and Interval exchanges



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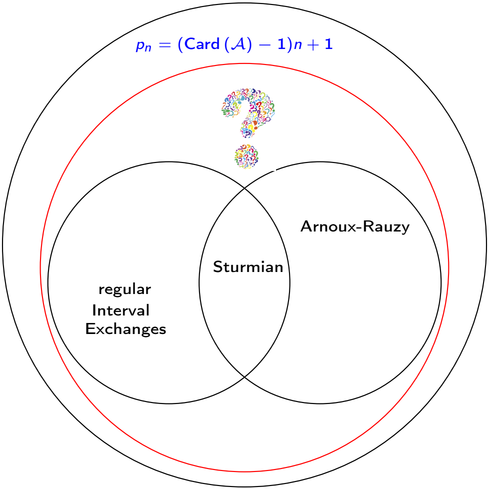

Extension graphs

The *extension graph* of a word $w \in \mathcal{L}$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

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\begin{aligned}
L(w) &= \{u \in \mathcal{A} \mid uw \in \mathcal{L}\} \\
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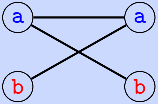
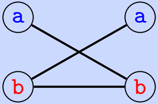
$\mathcal{E}(\varepsilon)$



$\mathcal{E}(\mathrm{a})$



$\mathcal{E}(\mathrm{b})$

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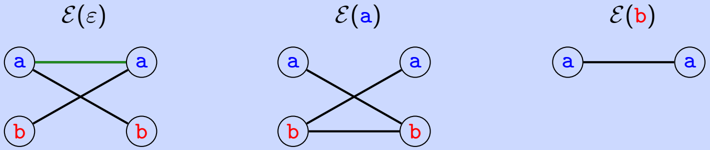

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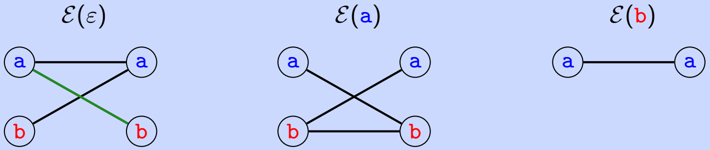

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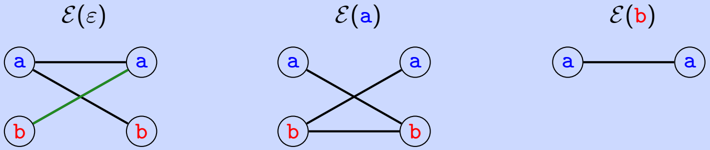

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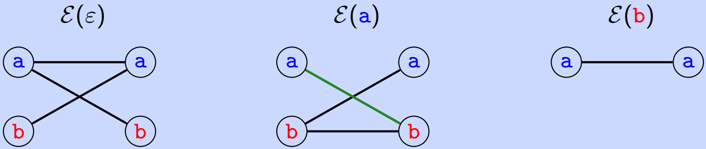

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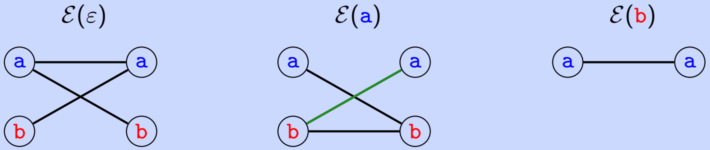

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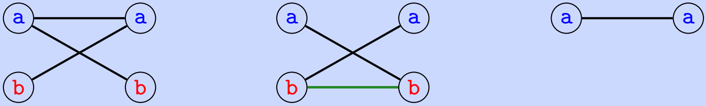
$\mathcal{E}(\varepsilon)$ $\mathcal{E}(\text{a})$ $\mathcal{E}(\text{b})$



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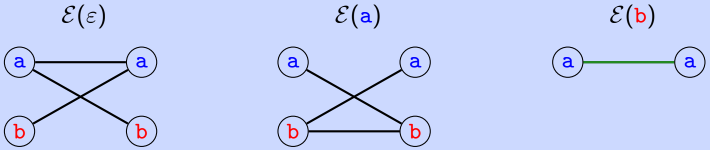
Example (Fibonacci, $\mathcal{L} = \{\varepsilon, \mathrm{a}, \mathrm{b}, \mathrm{aa}, \mathrm{ab}, \mathrm{ba}, \mathrm{aab}, \mathrm{aba}, \mathrm{baa}, \mathrm{bab}, \ldots\}$)



Extension graphs

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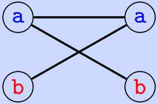
The *multiplicity* of a word w is the quantity

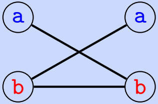
$$m(w) = \mathrm{Card}\,(B(w)) - \mathrm{Card}\,(L(w)) - \mathrm{Card}\,(R(w)) + 1.$$

Example (Fibonacci, $\mathcal{L} = \{\varepsilon, \mathrm{a}, \mathrm{b}, \mathrm{aa}, \mathrm{ab}, \mathrm{ba}, \mathrm{aab}, \mathrm{aba}, \mathrm{baa}, \mathrm{bab}, \ldots\}$)

$\mathcal{E}(\varepsilon)$



$\mathcal{E}(\mathrm{a})$



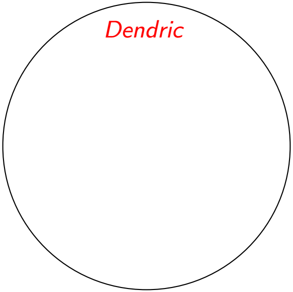
$\mathcal{E}(\mathrm{b})$

$m(\mathrm{a}) = 3 - 2 - 2 + 1 = 0$

Dendric and neutral languages

Definition

A language $\mathcal{L}$ is called (purely) *dendric* if the graph $\mathcal{E}(w)$ is a tree for any $w \in \mathcal{L}$.



Dendric and neutral languages

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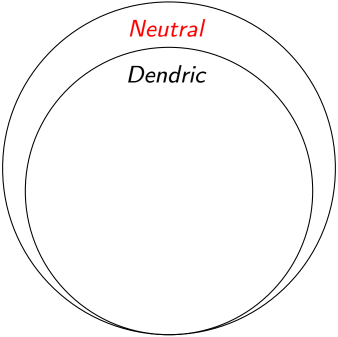

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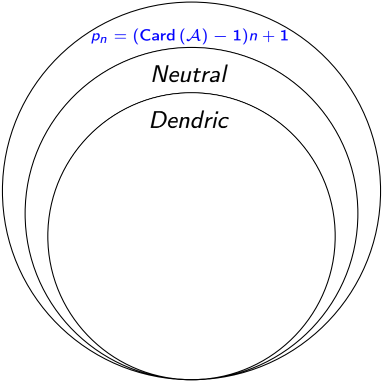

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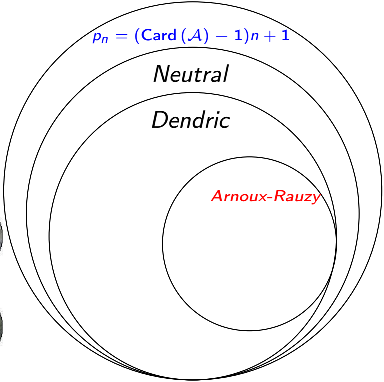

[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone: "**Acyclic, connected and tree sets**" (2014).]

Dendric and neutral languages

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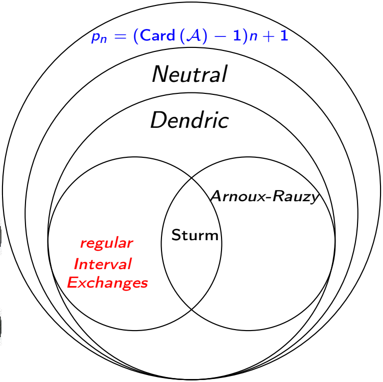


[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone: "Bifix codes and interval exchanges" (2015).]

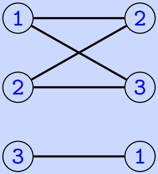
Dendric and neutral languages

Example (neutral not dendric)

The language of the fixed point $\tau(\sigma^\omega(\mathtt{a}))$ is a (recurrent) neutral language but it is not dendric (not acyclic).

$$\sigma : \begin{cases} \mathtt{a} \mapsto \mathtt{ab} \\ \mathtt{b} \mapsto \mathtt{cda} \\ \mathtt{c} \mapsto \mathtt{cd} \\ \mathtt{d} \mapsto \mathtt{abc} \end{cases} \qquad \tau : \begin{cases} \mathtt{a} \mapsto \mathtt{12} \\ \mathtt{b} \mapsto \mathtt{2} \\ \mathtt{c} \mapsto \mathtt{3} \\ \mathtt{d} \mapsto \mathtt{3} \end{cases}$$

$\mathcal{E}(\varepsilon)$

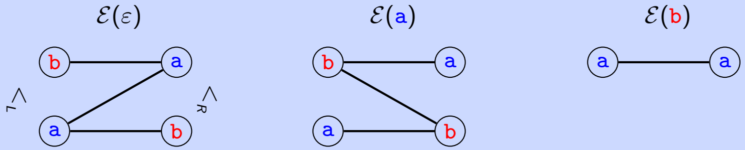


Planar dendric languages

Let $<_L$ and $<_R$ be two orders on $\mathcal{A}$.
For a language $\mathcal{L}$ and a word $w \in \mathcal{L}$, the graph $\mathcal{E}(w)$ is *compatible* with $<_L$ and $<_R$ if for any $(a, b), (c, d) \in B(w)$, one has

$$a <_L c \quad \Longrightarrow \quad b \leq_R d.$$

Example (Fibonacci, b $<_L$ a and a $<_R$ b)

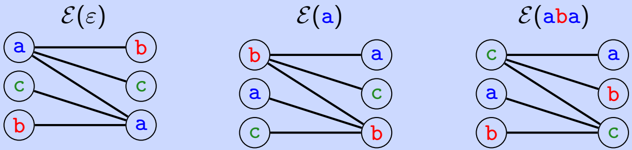


A biextendable language $\mathcal{L}$ is a *planar dendric language* w.r.t. $<_L$ and $<_R$ on $\mathcal{A}$ if for any $w \in \mathcal{L}$ the graph $\mathcal{E}(w)$ is a tree compatible with $<_L$ and $<_R$.

Planar dendric languages

Example

The *Tribonacci language* is **not** a planar dendric language.
Indeed, let us consider the extension graphs of the bispecial words ε, a and aba.



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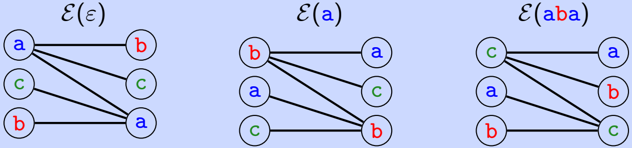
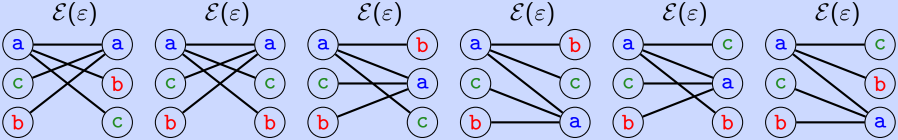


- $a <_L c <_L b$



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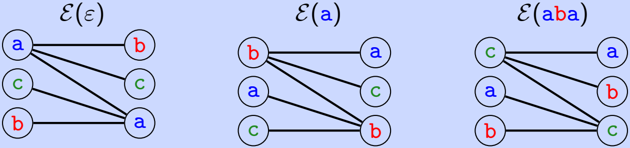
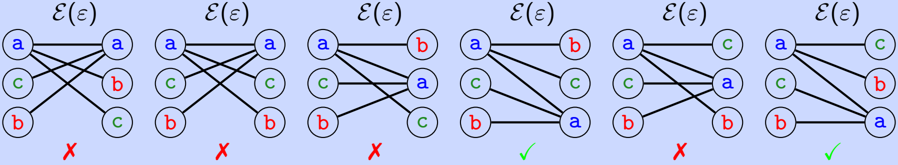


- a $<_L$ c $<_L$ b $\Longrightarrow$ b $<_R$ c $<_R$ a or c $<_R$ b $<_R$ a



Planar dendric languages

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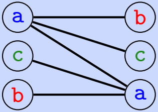
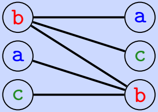
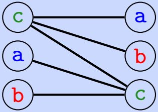
$\mathcal{E}(\varepsilon)$



$\mathcal{E}(\mathtt{a})$

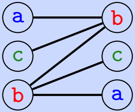


$\mathcal{E}(\mathtt{aba})$

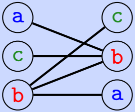


- a $<_L$ c $<_L$ b $\implies$ b $<_R$ c $<_R$ a or c $<_R$ b $<_R$ a

$\mathcal{E}(\mathtt{a})$



$\mathcal{E}(\mathtt{a})$



Planar dendric languages

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The *Tribonacci language* is **not** a planar dendric language.
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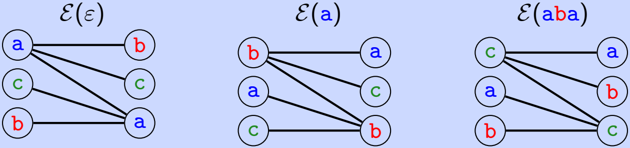

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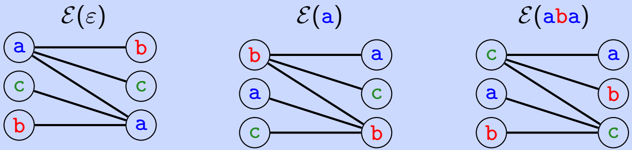
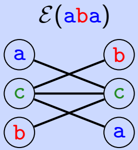


- $\underline{a <_L c <_L b}$ $\Longrightarrow$ $b <_R c <_R a$



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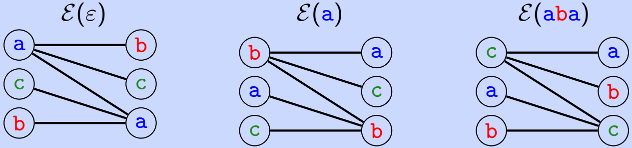
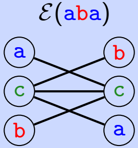


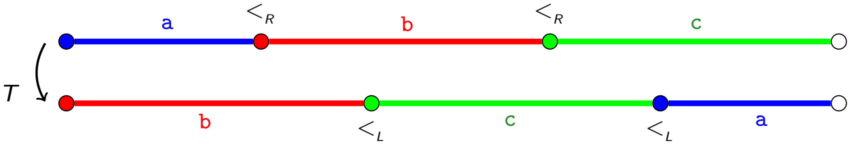
- $\underline{a <_L c <_L b} \quad \Longrightarrow \quad b <_R c <_R a$ ↯

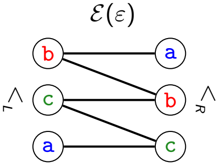


Planar dendric languages

Theorem [S. Ferenczi, L. Zamboni (2008)]

A set $\mathcal{L}$ is a regular interval exchange language if and only if it is a *recurrent planar dendric language*.





Recurrence and uniform recurrence

Definition

A language $\mathcal{L}$ is *recurrent* if for every $u, v \in \mathcal{L}$ there is a $w \in \mathcal{L}$ such that uwv is in $\mathcal{L}$.

Example (Fibonacci)

$$\mathbf{x} = \mathtt{abaababaabaababaababaabaababaaba}\cdots$$

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$\mathcal{L}$ is *uniformly recurrent* if for every $u \in \mathcal{L}$ there exists an $n \in \mathbb{N}$ such that u is a factor of every word of length n in $\mathcal{L}$.

Example (Fibonacci)

$$\mathbf{x} = \underbrace{\mathtt{abaa}}_{4}\,\mathtt{ba}\,\underbrace{\mathtt{baab}}_{4}\,\underbrace{\mathtt{aaba}}_{4}\,\mathtt{baababaaba}\,\underbrace{\mathtt{abab}}_{4}\,\mathtt{a}\cdots$$

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- Arnoux-Rauzy
- regular Interval Exchanges

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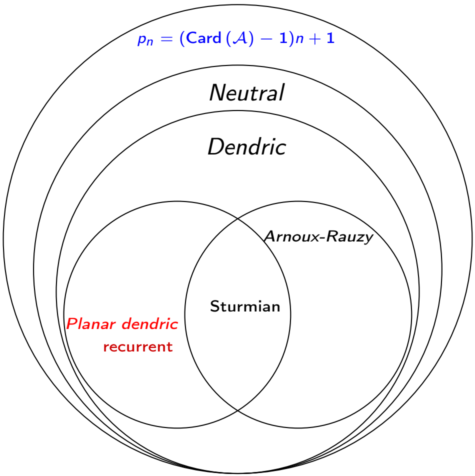
Proposition

Uniform recurrence $\Longrightarrow$ Recurrence.

Dendric and neutral languages



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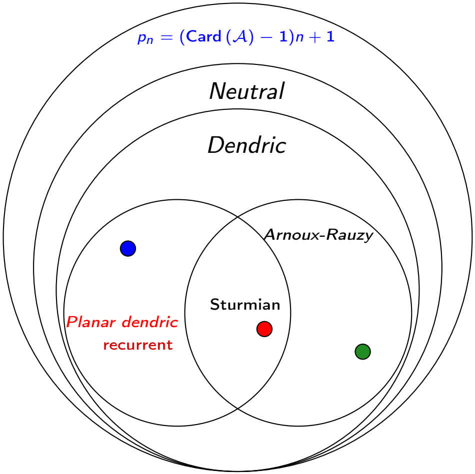
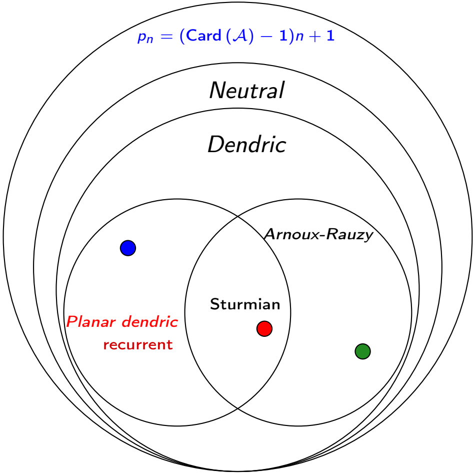


- Fibonacci
- Tribonacci
- regular IE

Dendric and neutral languages



- • Fibonacci
- ? 2-coded Fibonacci
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- ? 2-coded Tribonacci
- • regular IE
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Bifix codes

Definition

A *bifix code* is a set $B \subset \mathcal{A}^+$ of nonempty words that does not contain any proper prefix or suffix of its elements.

Example

- ✓ {aa, ab, ba}
- ✓ {aa, ab, bba, bbb}
- ✓ {ac, bcc, bcbca}
- ✗ { pivnice, pivo, pivovar }
- ✗ { becherovka, beton, rovka }
- ✗ { s, slivovice, vice }

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A bifix code $B \subset \mathcal{L}$ is *$\mathcal{L}$-maximal* if it is not properly contained in a bifix code $C \subset \mathcal{L}$.

Example (Fibonacci)

The set $B = \{\mathtt{aa}, \mathtt{ab}, \mathtt{ba}\}$ is an $\mathcal{L}$-maximal bifix code.
It is not an $\mathcal{A}^*$-maximal bifix code, since $B \subset B \cup \{\mathtt{bb}\}$.

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A *coding morphism* for a bifix code $B \subset A^+$ is a morphism $f : \mathcal{B}^* \to \mathcal{A}^*$ which maps bijectively $\mathcal{B}$ onto B.

Example

The map $f : \{\mathtt{u}, \mathtt{v}, \mathtt{w}\}^* \to \{\mathtt{a}, \mathtt{b}\}^*$ is a coding morphism for $B = \{\mathtt{aa}, \mathtt{ab}, \mathtt{ba}\}$.

$$f : \begin{cases} \mathtt{u} \mapsto \mathtt{aa} \\ \mathtt{v} \mapsto \mathtt{ab} \\ \mathtt{w} \mapsto \mathtt{ba} \end{cases}$$

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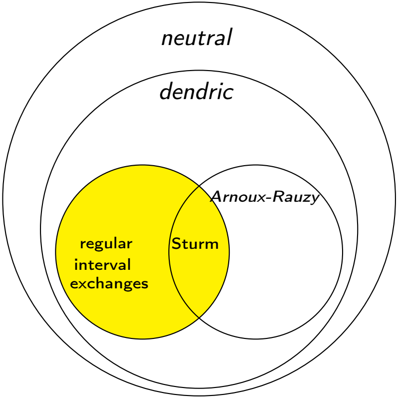
When $\mathcal{L}$ is factorial and B is an $\mathcal{L}$-maximal bifix code, the set $f^{-1}(\mathcal{L})$ is called a *maximal bifix decoding* of $\mathcal{L}$.

Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014)]

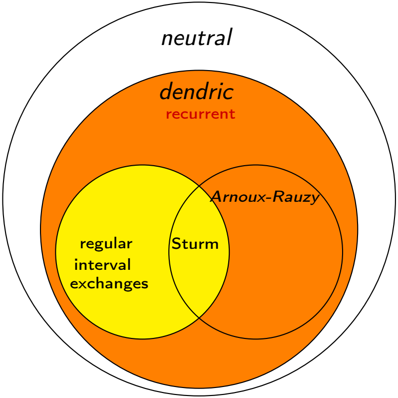
The family of regular interval exchanges languages is closed under maximal bifix decoding.



Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015)]

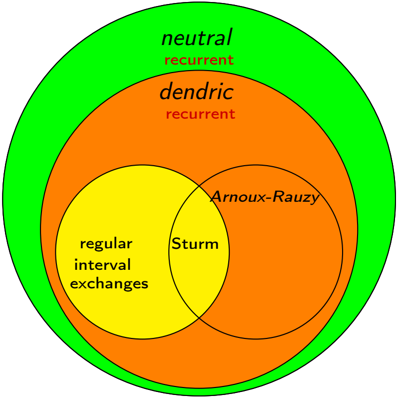
The family of *recurrent* dendric languages is closed under maximal bifix decoding.



Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015); D., Perrin (2016)]

The family of *recurrent* neutral languages is closed under maximal bifix decoding.



Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015); D., Perrin (2016, 2019)]

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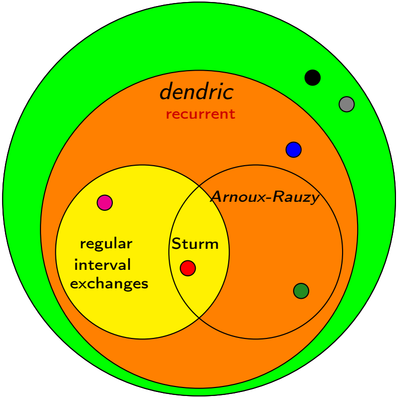

- Fibonacci
- 2-coded Fibonacci
- Tribonacci
- 2-coded Tribonacci

Return words

A (*right*) *return word* to w in $\mathcal{L}$ is a nonempty word u such that $wu \in \mathcal{L}$ starts and ends with w but has no w as an internal factor. Formally,

$$\mathcal{R}(w) = \{u \in \mathcal{A}^+ \mid wu \in \mathcal{L} \cap (\mathcal{A}^+ w \setminus \mathcal{A}^+ w \mathcal{A}^+)\}$$

Example (Fibonacci)

$\mathcal{R}(\mathrm{b}) = \{\mathrm{a\underline{b}}, \mathrm{aa\underline{b}}\}$

$$\varphi(\mathrm{a})^\omega = \mathrm{abaababaabaababaababaabababaabaab}\cdots$$

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Example (Fibonacci)

$\mathcal{R}(\mathtt{aa}) = \{\mathtt{b}\underline{\mathtt{aa}}, \mathtt{bab}\underline{\mathtt{aa}}\}$

$$\varphi(\mathtt{a})^\omega = \mathtt{abaababaabaababaababaabaababaabaab}\cdots$$

Cardinality of return words

Theorem [Vuillon (2001)]

Let $\mathcal{L}$ be a Sturmian language. For every $w \in \mathcal{L}$, one has

$$\mathrm{Card}\,(\mathcal{R}(w)) = 2.$$

Cardinality of return words

Theorem [Vuillon (2001); Balková, Pelantová, Steiner (2008)]

Let $\mathcal{L}$ be a recurrent neutral language. For every $w \in \mathcal{L}$, one has

$$\text{Card}\,(\mathcal{R}(w)) = \text{Card}\,(\mathcal{A})\,.$$

Cardinality of return words

Theorem [Vuillon (2001); Balková, Pelantová, Steiner (2008)]

Let $\mathcal{L}$ be a . For every $w \in \mathcal{L}$, one has

$$\text{Card}\,(\mathcal{R}(w)) = \text{Card}\,(\mathcal{A})\,.$$

Corollary

A neutral (dendric) language is recurrent if and only if it is uniformly recurrent

Proof. A recurrent language $\mathcal{L}$ is uniformly recurrent if and only if $\mathcal{R}(w)$ is finite for all $w \in \mathcal{L}$.

Rauzy graphs
and Stallings foldings

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2015),]

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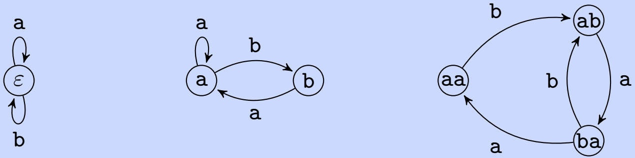

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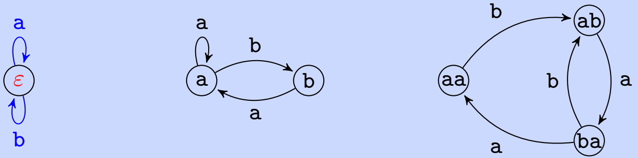

$$\Gamma_{\varepsilon} = \langle \mathrm{a}, \mathrm{b} \rangle = \mathbb{F}_{\mathcal{A}}$$

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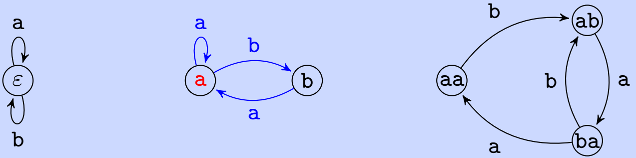

$\Gamma_{\mathtt{a}} = \langle \mathtt{a}, \mathtt{ba} \rangle$

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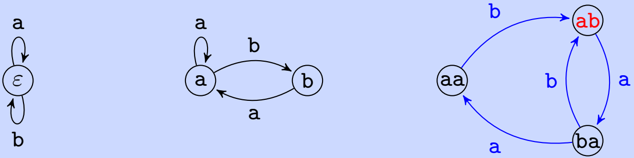

$$\Gamma_{\mathtt{ab}} = \langle \mathtt{a}(\mathtt{ba})^{*}\mathtt{ab} \rangle$$

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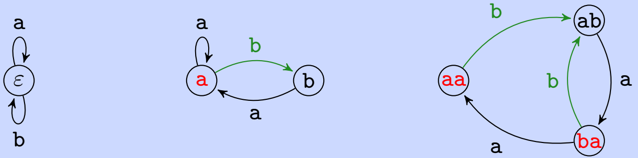

$G_2(\mathcal{L}) \rightsquigarrow G_1(\mathcal{L})$

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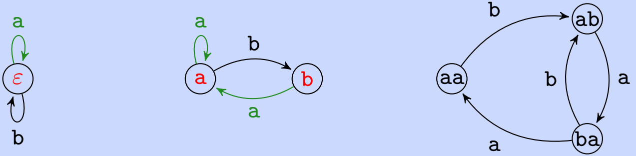

$$G_2(\mathcal{L}) \rightsquigarrow G_1(\mathcal{L}) \rightsquigarrow G_0(\mathcal{L})$$

The Return Theorem

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Ridone (2015)]

Let $\mathcal{L}$ be a recurrent dendric language. For every $w \in \mathcal{L}$, $\mathcal{R}(w)$ is a basis of the free group $\mathbb{F}_{\mathcal{A}}$.

Example (Fibonacci)

The set $\mathcal{R}(\mathtt{b}) = \{\mathtt{ab}, \mathtt{aab}\}$ is a basis of the free group. Indeed,

$$\begin{aligned} \mathtt{a} &= \mathtt{aab}\,(\mathtt{ab})^{-1} \\ \mathtt{b} &= \mathtt{a}^{-1}\,\mathtt{ab} \end{aligned}$$

The Return Theorem

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Ridone (2015)]

Let $\mathcal{L}$ be a recurrent dendric language. For every $w \in \mathcal{L}$, $\mathcal{R}(w)$ is a basis of the free group $\mathbb{F}_{\mathcal{A}}$.

Example (Fibonacci)

The set $\mathcal{R}(\mathtt{aa}) = \{\mathtt{aab}, \mathtt{aabab}\}$ is a basis of the free group. Indeed,

$$\begin{aligned} \mathtt{a} &= \mathtt{aab}\,(\mathtt{aabab})^{-1}\,\mathtt{aab} \\ \mathtt{b} &= \mathtt{a}^{-1}\,\mathtt{a}^{-1}\,\mathtt{aab} \end{aligned}$$

Děkuji za pozornost*

* Ďakujem za pozornosť

Merci pour l'attention