Palindromes and Tree Sets

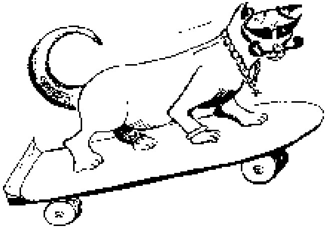
Francesco Dolce

LaCIM UQÀM

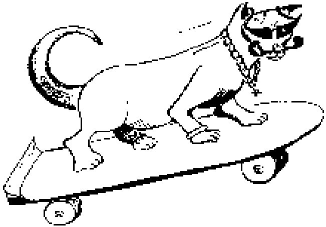
Atelier
"Combinatoire des mots et pavages"

"Combinatorics on Words and Tilings" Workshop

Montréal, 4 avril 2017



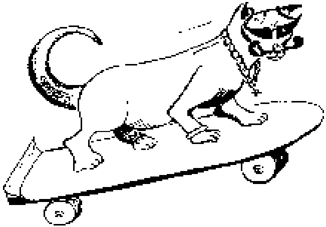
GoFlowolFoG



GoFlowolFoG

«You can summon him by trying to take on his characteristics - relaxing, fantasising that you're 'cool', and letting go of your frustration momentarily. Visualise him zipping along on his skateboard, accompanied by a slight breeze and his Mantra: 'Neeeoooow'.»

[philhine.org.uk]



GoFlowolFoG

«*You can summon him by trying to take on his characteristics - relaxing, fantasising that you're 'cool', and letting go of your frustration momentarily. Visualise him zipping along on his skateboard, accompanied by a slight breeze and his Mantra: 'Neeeoooow'.*»

«*We decided that the 'name' of the Spirit would* [..] *be* Go Flow. *This was mirrored to give the name* GoFlowolFoG *- which sounds suitably 'magical'.*»

[philhine.org.uk]

Palindromes

A *palindrome* is a word $w = \tilde{w}$ as, for instance:

 non, esse, aveva, rossor, ottetto, . . .

Palindromes

A *palindrome* is a word $w = \tilde{w}$ as, for instance:

-  non, esse, aveva, rossor, ottetto, ...
-  eye, noon, sagas, racecar, ...
-  ici, été, coloc, kayak, ...

Palindromes

A *palindrome* is a word $w = \tilde{w}$ as, for instance:

 non, esse, aveva, rossor, ottetto, . . .

 eye, noon, sagas, racecar, . . .

 ici, été, coloc, kayak, . . .

 saippuakivikauppias, . . .

Palindromes

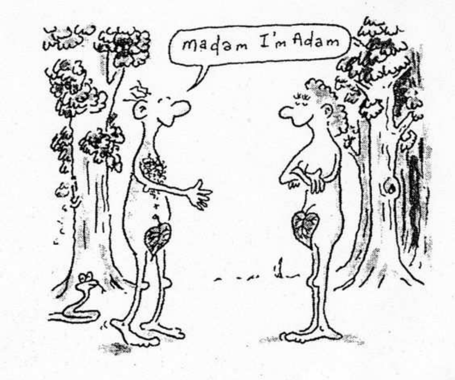
A *palindrome* is a word $w = \tilde{w}$ as, for instance:

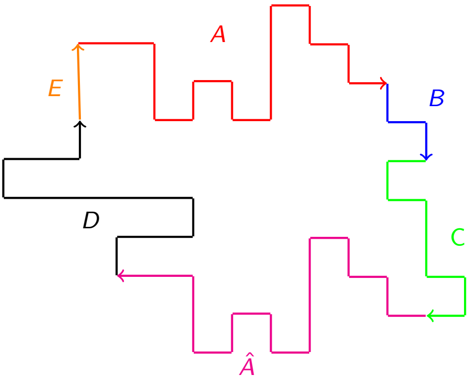
- non, esse, aveva, rossor, ottetto, . . .
- eye, noon, sagas, racecar, . . .
- ici, été, coloc, kayak, . . .
- saippuakivikauppias, . . .
- ojo, somos, reconocer, . . .
- Krk, potop, ići, . . .
- топот, довод, кабак,
- وَلَو ، وَدُ ، مُهِمّ ، أَبَا ، . . .
- À Laval elle l'avala, . . .

Palindromes

A *palindrome* is a word $w = \tilde{w}$ as, for instance:

- non, esse, aveva, rossor, ottetto, . . .
- eye, noon, sagas, racecar, . . .
- ici, été, coloc, kayak, . . .
- saippuakivikauppias, . . .
- ojo, somos, reconocer, . . .
- Krk, potop, ići, . . .
- топот, довод, кабак,
- وَلَو , وَدُّ , مُهِمّ , أَبَا , . . .
- À Laval elle l'avala, . . .

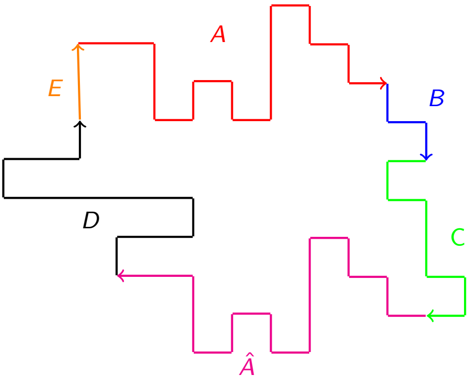




Conway's Criterion: B, C, D, E palindromes.

$$B = \downarrow\rightarrow\downarrow, \qquad C = \leftarrow\downarrow\rightarrow\downarrow\downarrow\rightarrow\downarrow\leftarrow,$$

$$D = \uparrow\rightarrow\rightarrow\uparrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\uparrow\rightarrow\rightarrow\uparrow, \qquad E = \uparrow\uparrow.$$

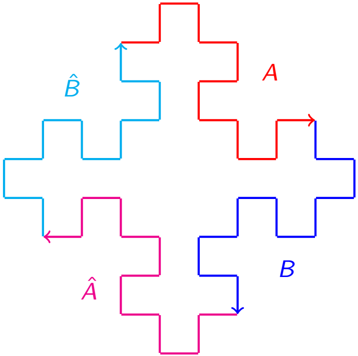


Conway's Criterion: B, C, D, E palindromes.

$$B = 303, \qquad C = 23033032,$$
$$D = 1001333331001, \qquad E = 11.$$

Theorem [A. Blondin-Massé, A. Garon, S. Labbé (2013)]

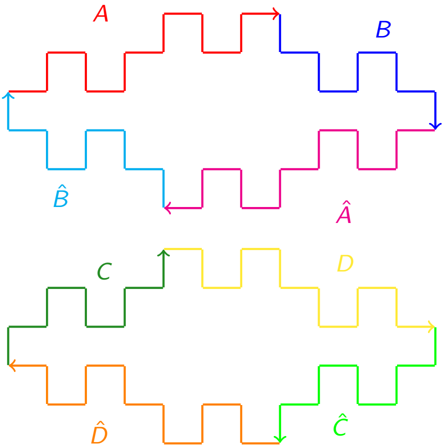
If $AB\hat{A}\hat{B}$ is a BN-factorisation of a Fibonacci tile, then A and B are palindromes.



$A = 0103032303010$, $B = 3032321232303$,

Theorem [A. Blondin-Massé, S. Brlek, A. Garon, S. Labbé (2009)]

If $AB\hat{A}\hat{B}$ and $CD\hat{C}\hat{D}$ are the BN-factorisation of a prime double square, then A, B, C, D are palindromes.



Full words

Theorem [X. Droubay, J. Justin, G. Pirillo (2001)]

A word of length n has at most $n+1$ palindrome factors

A word with maximal number of palindromes is *full* (or *rich*).

Full words

Theorem [X. Droubay, J. Justin, G. Pirillo (2001)]

A word of length n has at most $n+1$ palindrome factors

A word with maximal number of palindromes is *full* (or *rich*).

Example

- TRUMP, PUTIN, LE PEN, FILLON are rich.
- TRUDEAU, MERKEL, GENTILONI, MÉLENCHON are not rich.

Full words

Theorem [X. Droubay, J. Justin, G. Pirillo (2001)]

A word of length n has at most $n+1$ palindrome factors

A word with maximal number of palindromes is *full* (or *rich*).

Example

- TRUMP, PUTIN, LE PEN, FILLON are rich.
- TRUDEAU, MERKEL, GENTILONI, MÉLENCHON are not rich.

$$|\text{FRANÇOIS}| = 8 \quad \text{and} \quad \mathrm{Card}\left(\{\varepsilon, \text{F}, \text{R}, \text{A}, \text{N}, \text{Ç}, \text{O}, \text{I}, \text{S}\}\right) = 9 = 8+1$$

$$|\text{PENELOPE}| = 8 \quad \text{and} \quad \mathrm{Card}\left(\{\varepsilon, \text{P}, \text{E}, \text{N}, \text{L}, \text{O}, \text{ENE}\}\right) = 7 < 8+1$$

Full words

Theorem [X. Droubay, J. Justin, G. Pirillo (2001)]

A word of length n has at most $n+1$ palindrome factors

A word with maximal number of palindromes is *full* (or *rich*).
A factorial set is *full* if all its elements are full.

Example (Fibonacci)

Let S be the set of factors of the fixed-point $\varphi^\omega(0)$ of

$$\varphi : 0 \mapsto 01, \quad 1 \mapsto 0.$$

Every word $w \in S$ is full. For instance,

$$\mathrm{Pal}(01001) = \{\varepsilon, 0, 1, 00, 010, 1001\}.$$

Arnoux-Rauzy sets

Definition

An *Arnoux-Rauzy* set is a factorial set **closed under reversal** with $p_n = (\mathrm{Card}\,(A) - 1)n + 1$ having a unique right special factor for each length.

Examples

- Fibonacci: factors of the fixed-point $\varphi^\omega(0)$, where $\varphi : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases}$.
- Tribonacci: factors of the fixed-point $\psi^\omega(0)$, where $\psi : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 02 \\ 2 \mapsto 0 \end{cases}$.

Arnoux-Rauzy sets

Definition

An *Arnoux-Rauzy* set is a factorial set **closed under reversal** with $p_n = (\text{Card}\,(A) - 1)n + 1$ having a unique right special factor for each length.

Examples

- Fibonacci: factors of the fixed-point $\varphi^\omega(0)$, where $\varphi : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases}$.
- Tribonacci: factors of the fixed-point $\psi^\omega(0)$, where $\psi : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 02 \\ 2 \mapsto 0 \end{cases}$.

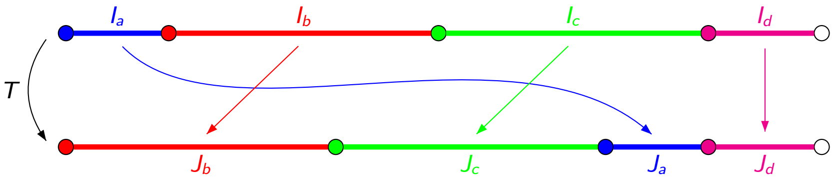
Theorem [X. Droubay, J. Justin, G. Pirillo (2001)]

Arnoux-Rauzy sets are full.

Interval exchanges

Let $(I_\alpha)_{\alpha \in A}$ and $(J_\alpha)_{\alpha \in A}$ be two partitions of a semi-interval I.
An *interval exchange transformation* (IET) is a map $T : I \to I$ defined by

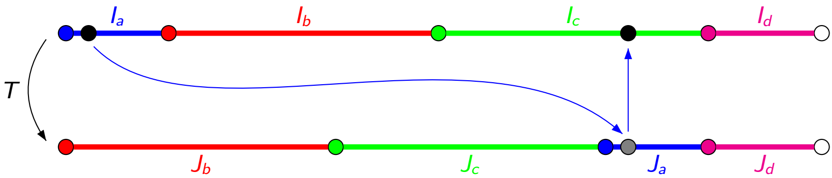
$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha.$$



Interval exchanges

Let $(I_\alpha)_{\alpha \in A}$ and $(J_\alpha)_{\alpha \in A}$ be two partitions of a semi-interval I.
An *interval exchange transformation* (IET) is a map $T : I \to I$ defined by

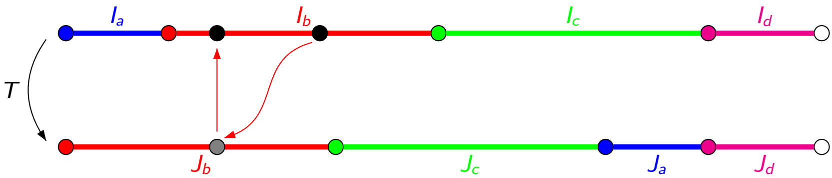
$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha.$$



Interval exchanges

Let $(I_\alpha)_{\alpha \in A}$ and $(J_\alpha)_{\alpha \in A}$ be two partitions of a semi-interval I.
An *interval exchange transformation* (IET) is a map $T : I \to I$ defined by

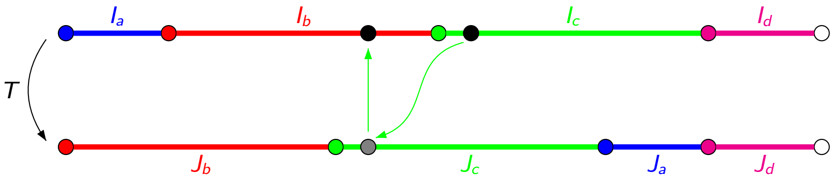
$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha .$$



Interval exchanges

Let $(I_\alpha)_{\alpha \in A}$ and $(J_\alpha)_{\alpha \in A}$ be two partitions of a semi-interval I.
An *interval exchange transformation* (IET) is a map $T : I \to I$ defined by

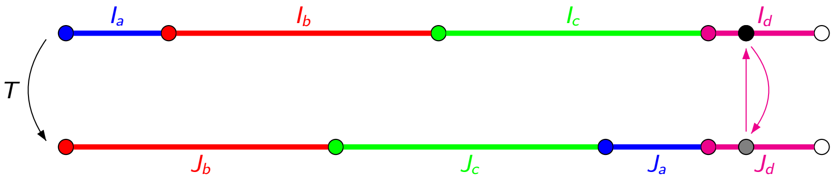
$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha .$$



Interval exchanges

Let $(I_\alpha)_{\alpha \in A}$ and $(J_\alpha)_{\alpha \in A}$ be two partitions of a semi-interval I.
An *interval exchange transformation* (IET) is a map $T : I \to I$ defined by

$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha.$$



Interval exchanges

T is *minimal* if for any point $z \in I$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in I.

T is *regular* if the orbits of the separation points are infinite and disjoint.

Theorem [M. Keane (1975)]

A regular interval exchange transformation is minimal.

Interval exchanges

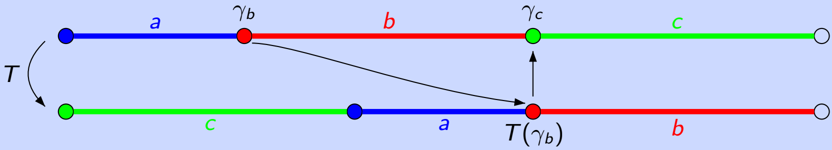
T is *minimal* if for any point $z \in I$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in I.

T is *regular* if the orbits of the separation points are infinite and disjoint.

Theorem [M. Keane (1975)]

A regular interval exchange transformation is minimal.

Example (the converse is not true)

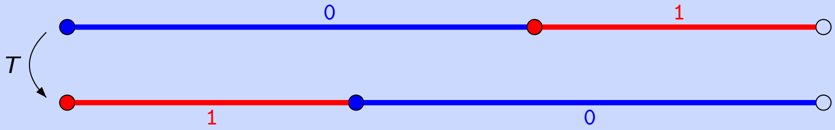


Interval exchanges

The *natural coding* of T relative to $z \in I$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in A^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha.$$

Example (Fibonacci, $z = (3 - \sqrt{5})/2$)

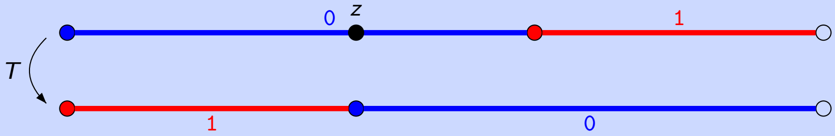


Interval exchanges

The *natural coding* of T relative to $z \in I$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in A^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha.$$

Example (Fibonacci, $z = (3 - \sqrt{5})/2$)



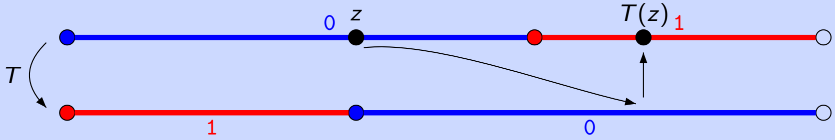
$\Sigma_T(z) = 0$

Interval exchanges

The *natural coding* of T relative to $z \in I$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in A^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha.$$

Example (Fibonacci, $z = (3 - \sqrt{5})/2$)



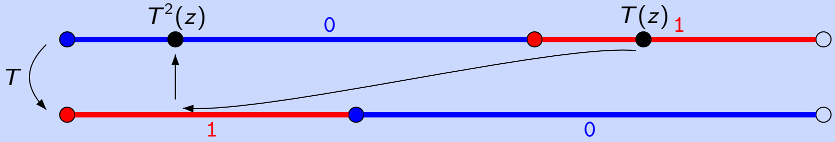
$\Sigma_T(z) = 0\,1$

Interval exchanges

The *natural coding* of T relative to $z \in I$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in A^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha .$$

Example (Fibonacci, $z = (3 - \sqrt{5})/2$)



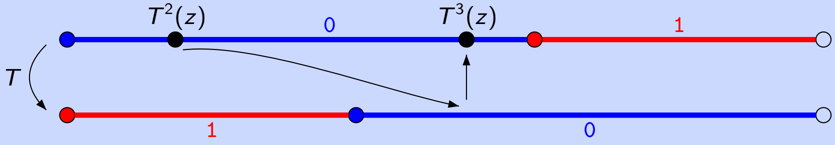
$\Sigma_T(z) = 0\,1\,0$

Interval exchanges

The *natural coding* of T relative to $z \in I$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in A^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha.$$

Example (Fibonacci, $z = (3-\sqrt{5})/2$)



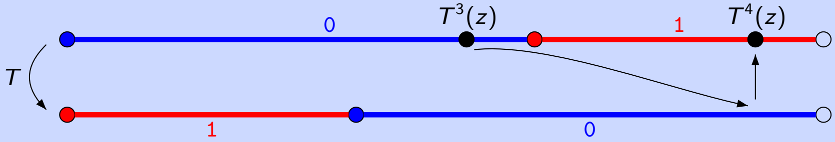
$\Sigma_T(z) = 0\,1\,0\,0$

Interval exchanges

The *natural coding* of T relative to $z \in I$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in A^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha.$$

Example (Fibonacci, $z = (3 - \sqrt{5})/2$)



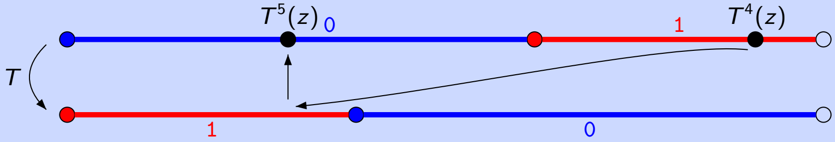
$\Sigma_T(z) = 0\,1\,0\,0\,1$

Interval exchanges

The *natural coding* of T relative to $z \in I$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in A^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha .$$

Example (Fibonacci, $z = (3 - \sqrt{5})/2$)



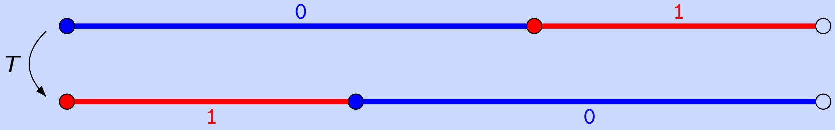
$$\Sigma_T(z) = 0\,1\,0\,0\,1\,0\cdots$$

Interval exchanges

The set $\mathcal{L}(T) = \bigcup_{z \in I} \text{Fac}(\Sigma_T(z))$ is said a (*minimal*, *regular*) *interval exchange set*.

Remark. If T is minimal, $\text{Fac}(\Sigma_T(z))$ does not depend on the point z.

Example (Fibonacci)



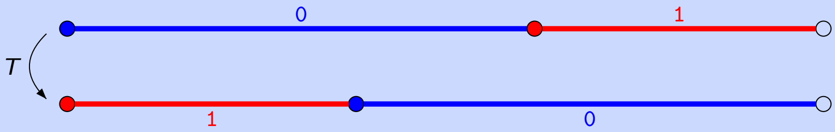
$$\mathcal{L}(T) = \left\{\varepsilon, 0, 1, 00, 01, 10, 001, 010, 100, \dots\right\}$$

Interval exchanges

The set $\mathcal{L}(T) = \bigcup_{z \in I} \mathrm{Fac}\,(\Sigma_T(z))$ is said a (*minimal*, *regular*) *interval exchange set*.

Remark. If T is minimal, $\mathrm{Fac}\,(\Sigma_T(z))$ does not depend on the point z.

Example (Fibonacci)



$$\mathcal{L}(T) = \left\{\varepsilon, 0, 1, 00, 01, 10, 001, 010, 100, \ldots\right\}$$

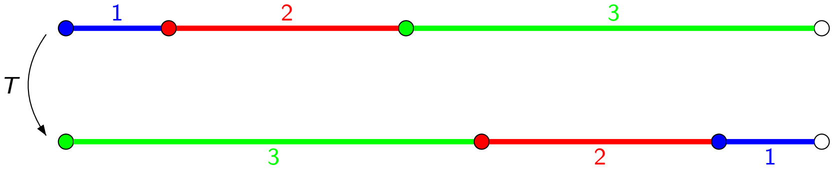
Proposition

Regular interval exchange sets have factor complexity $p_n = (\mathrm{Card}\,(A) - 1)n + 1$.

Interval exchanges

Theorem [P. Baláži, Z. Masáková, E. Pelantová (2007)]

Regular interval exchange sets **closed under reverse** are full.



T closed under reverse $\iff$ $\pi = (n \;\; n-1 \;\; \cdots \;\; 2 \;\; 1)$

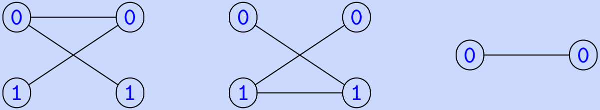
Extension graphs

The *extension graph* of a word $w \in S$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{a \in A \,|\, aw \in S\}, \\ R(w) &= \{a \in A \,|\, wa \in S\}, \\ B(w) &= \{(a,b) \in A \,|\, awb \in S.\} \end{aligned}$$

Example (Fibonacci, $S = \{\varepsilon, 0, 1, 00, 01, 10, 001, 010, 100, 101, \ldots\}$)

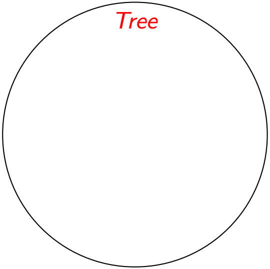
$\mathcal{E}(\varepsilon)$ $\mathcal{E}(0)$ $\mathcal{E}(1)$



Tree sets

Definition

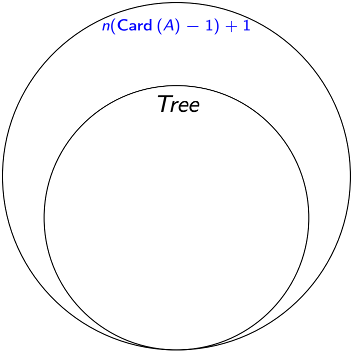
A factorial set S is called a *tree set* (of characteristic 1) if $\mathcal{E}(w)$ is a tree for any $w \in S$.



Tree sets

Definition

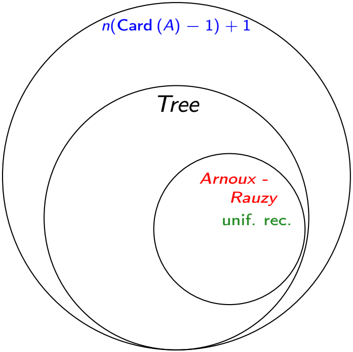
A factorial set S is called a *tree set* (of characteristic 1) if $\mathcal{E}(w)$ is a tree for any $w \in S$.



Tree sets

Definition

A factorial set S is called a *tree set* (of characteristic 1) if $\mathcal{E}(w)$ is a tree for any $w \in S$.

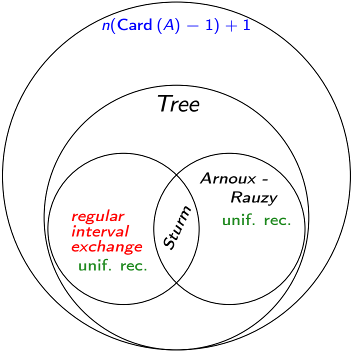


[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014)]

Tree sets

Definition

A factorial set S is called a *tree set* (of characteristic 1) if $\mathcal{E}(w)$ is a tree for any $w \in S$.

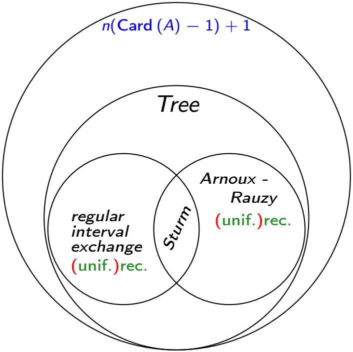


[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

Tree sets

Definition

A factorial set S is called a *tree set* (of characteristic 1) if $\mathcal{E}(w)$ is a tree for any $w \in S$.

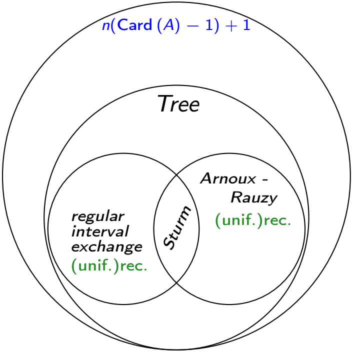


[D., Perrin (2016)]

Tree sets

Definition

A factorial set S is called a *tree set* (of characteristic 1) if $\mathcal{E}(w)$ is a tree for any $w \in S$.



Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2016)]

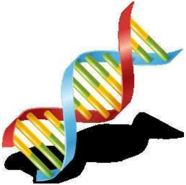
A (uniformly) recurrent tree set **closed under reversal** is full.

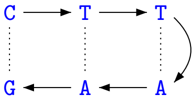
σ-*palindromes*

Let σ be an antimorphism.
A word w is a *σ-palindrome* if $w = \sigma(w)$.

Example

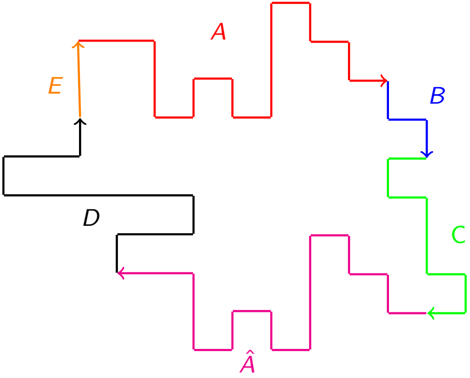
Let $\sigma : \mathtt{A} \leftrightarrow \mathtt{T},\ \mathtt{C} \leftrightarrow \mathtt{G}$.
The word CTTAAG is a σ-palindrome.





σ-palindromes

Let σ be an antimorphism.
A word w is a *σ-palindrome* if $w = \sigma(w)$.



$\hat{} = \sigma \; : \; 0 \leftrightarrow 2 \quad 1 \leftrightarrow 3$

$A = 0033010301110303 0$

$\hat{A} = 2121233321212112 2$

σ-*palindromes*

Let σ be an antimorphism.
A word w is a *σ-palindrome* if $w = \sigma(w)$.

Theorem [Š. Starosta (2011)]

$\text{Card}\,(\text{Pal}_\sigma(w)) \leq |w| + 1 - \gamma_\sigma(w)$ with $\gamma_\sigma(w) = \#$ transposition acting on w.

A word (resp. set) is *σ-full* if the equality holds (resp. for all its elements).

σ-palindromes

Let σ be an antimorphism.
A word w is a *σ-palindrome* if $w = \sigma(w)$.

Theorem [Š. Starosta (2011)]

$\mathrm{Card}\,(\mathrm{Pal}_\sigma(w)) \leq |w| + 1 - \gamma_\sigma(w)$ with $\gamma_\sigma(w) = \#$ transposition acting on w.

A word (resp. set) is *σ-full* if the equality holds (resp. for all its elements).

Example

Let $\sigma : \mathtt{I} \leftrightarrow \mathtt{M},\ \mathtt{O} \leftrightarrow \mathtt{T}$ and $\tau = \mathtt{J} \leftrightarrow \mathtt{O},\ \mathtt{K} \leftrightarrow \mathtt{R}$, fixing all other letters.

σ-palindromes

Let σ be an antimorphism.
A word w is a *σ-palindrome* if $w = \sigma(w)$.

Theorem [Š. Starosta (2011)]

$\mathrm{Card}\,(\mathrm{Pal}_\sigma(w)) \leq |w| + 1 - \gamma_\sigma(w)$ with $\gamma_\sigma(w) = \#$ transposition acting on w.

A word (resp. set) is *σ-full* if the equality holds (resp. for all its elements).

Example

Let $\sigma : \mathtt{I} \leftrightarrow \mathtt{M},\ \mathtt{O} \leftrightarrow \mathtt{T}$ and $\tau = \mathtt{J} \leftrightarrow \mathtt{O},\ \mathtt{K} \leftrightarrow \mathtt{R}$, fixing all other letters.

$$\begin{aligned} \mathrm{Card}\,(\mathrm{Pal}_\sigma(\mathtt{TIMO})) &= \mathrm{Card}\,(\{\varepsilon,\ \mathtt{IM},\ \mathtt{TIMO}\}) \\ &= 3 = 4 + 1 - 2 \end{aligned}$$

σ-palindromes

Let σ be an antimorphism.
A word w is a *σ-palindrome* if $w = \sigma(w)$.

Theorem [Š. Starosta (2011)]

$\mathrm{Card}\,(\mathrm{Pal}_\sigma(w)) \leq |w| + 1 - \gamma_\sigma(w)$ with $\gamma_\sigma(w) = \#$ transposition acting on w.

A word (resp. set) is *σ-full* if the equality holds (resp. for all its elements).

Example

Let $\sigma : \mathtt{I} \leftrightarrow \mathtt{M},\ \mathtt{O} \leftrightarrow \mathtt{T}$ and $\tau = \mathtt{J} \leftrightarrow \mathtt{O},\ \mathtt{K} \leftrightarrow \mathtt{R}$, fixing all other letters.

$$\begin{aligned} \mathrm{Card}\,(\mathrm{Pal}_\sigma(\mathtt{TIMO})) &= \mathrm{Card}\,(\{\varepsilon,\ \mathtt{IM},\ \mathtt{TIMO}\}) \\ &= 3 = 4 + 1 - 2 \\ \mathrm{Card}\,(\mathrm{Pal}_\tau(\mathtt{JARKKO})) &= \mathrm{Card}\,(\{\varepsilon,\ \mathtt{A},\ \mathtt{RK}\}) \\ &= 3 < 5 = 6 + 1 - 2 \end{aligned}$$

G-*palindromes*

Let G be a group containing at least one antimorphism.
A word w is a *G-palindrome* if there exists a nontrivial $g \in G$ s.t. $w = g(w)$.

G-palindromes

Let G be a group containing at least one antimorphism.
A word w is a *G-palindrome* if there exists a nontrivial $g \in G$ s.t. $w = g(w)$.

Example

Let $G = \langle \sigma, \tau \rangle$ with $\sigma : \mathtt{A} \leftrightarrow \mathtt{E},\ \mathtt{I} \leftrightarrow \mathtt{V},\ \mathtt{R} \leftrightarrow \mathtt{X},\ \mathtt{O} \leftrightarrow \mathtt{L}$
and $\tau : \mathtt{A} \leftrightarrow \mathtt{J},\ \mathtt{L} \leftrightarrow \mathtt{S}$ fixing the other letters.

The following are G-palindromes:

G-palindromes

Let G be a group containing at least one antimorphism.
A word w is a *G-palindrome* if there exists a nontrivial $g \in G$ s.t. $w = g(w)$.

Example

Let $G = \langle \sigma, \tau \rangle$ with $\sigma : \mathtt{A} \leftrightarrow \mathtt{E},\ \mathtt{I} \leftrightarrow \mathtt{V},\ \mathtt{R} \leftrightarrow \mathtt{X},\ \mathtt{O} \leftrightarrow \mathtt{L}$
and $\tau : \mathtt{A} \leftrightarrow \mathtt{J},\ \mathtt{L} \leftrightarrow \mathtt{S}$ fixing the other letters.

The following are G-palindromes:

- XAVIER, fixed by σ,

G-*palindromes*

Let G be a group containing at least one antimorphism.
A word w is a G-*palindrome* if there exists a nontrivial $g \in G$ s.t. $w = g(w)$.

Example

Let $G = \langle \sigma, \tau \rangle$ with $\sigma : \mathtt{A} \leftrightarrow \mathtt{E},\ \mathtt{I} \leftrightarrow \mathtt{V},\ \mathtt{R} \leftrightarrow \mathtt{X},\ \mathtt{O} \leftrightarrow \mathtt{L}$
and $\tau : \mathtt{A} \leftrightarrow \mathtt{J},\ \mathtt{L} \leftrightarrow \mathtt{S}$ fixing the other letters.

The following are G-palindromes:

- `XAVIER`, fixed by σ,
- `ÉLISE`, fixed by τ,

G-palindromes

Let G be a group containing at least one antimorphism.
A word w is a *G-palindrome* if there exists a nontrivial $g \in G$ s.t. $w = g(w)$.

Example

Let $G = \langle \sigma, \tau \rangle$ with $\sigma : \mathtt{A} \leftrightarrow \mathtt{E},\ \mathtt{I} \leftrightarrow \mathtt{V},\ \mathtt{R} \leftrightarrow \mathtt{X},\ \mathtt{O} \leftrightarrow \mathtt{L}$
and $\tau : \mathtt{A} \leftrightarrow \mathtt{J},\ \mathtt{L} \leftrightarrow \mathtt{S}$ fixing the other letters.

The following are G-palindromes:

- XAVIER, fixed by σ,
- ÉLISE, fixed by τ,
- JOSÉ, fixed by $\sigma\tau\sigma$,

G-palindromes

Let G be a group containing at least one antimorphism.
A word w is a *G-palindrome* if there exists a nontrivial $g \in G$ s.t. $w = g(w)$.

Example

Let $G = \langle \sigma, \tau \rangle$ with $\sigma : \mathtt{A} \leftrightarrow \mathtt{E},\ \mathtt{I} \leftrightarrow \mathtt{V},\ \mathtt{R} \leftrightarrow \mathtt{X},\ \mathtt{O} \leftrightarrow \mathtt{L}$
and $\tau : \mathtt{A} \leftrightarrow \mathtt{J},\ \mathtt{L} \leftrightarrow \mathtt{S}$ fixing the other letters.

The following are G-palindromes:

- XAVIER, fixed by σ,
- ÉLISE, fixed by τ,
- JOSÉ, fixed by $\sigma\tau\sigma$,

while NADIA is fixed only by id.

G-*palindromes*

Let G be a group containing at least one antimorphism.
A word w is a G-*palindrome* if there exists a nontrivial $g \in G$ s.t. $w = g(w)$.

Example

Let $G = \langle \sigma, \tau \rangle$ with $\sigma : \mathtt{A} \leftrightarrow \mathtt{E},\ \mathtt{I} \leftrightarrow \mathtt{V},\ \mathtt{R} \leftrightarrow \mathtt{X},\ \mathtt{O} \leftrightarrow \mathtt{L}$
and $\tau : \mathtt{A} \leftrightarrow \mathtt{J},\ \mathtt{L} \leftrightarrow \mathtt{S}$ fixing the other letters.

The following are G-palindromes:

- XAVIER, fixed by σ,
- ÉLISE, fixed by τ,
- JOSÉ, fixed by $\sigma\tau\sigma$,

while NADIA is fixed only by id.

A word (set) is G-*full* if "the number of G-palindromes is maximal".

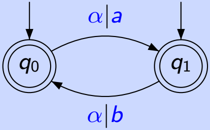
Doubling transducer

A *doubling transducer* is a transducer with set of states $\{q_0, q_1\}$ such that:

1. the input automata is a group automaton,
2. the output labels of the edges are all distinct.

Example

$\Sigma = \{\alpha\}$
$A = \{a, b\}$



Doubling transducer

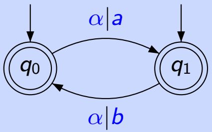
A *doubling transducer* is a transducer with set of states $\{q_0, q_1\}$ such that:

1. the input automata is a group automaton,
2. the output labels of the edges are all distinct.

$\delta_0, \delta_1 : \Sigma^* \to A^*$ are defined by $\delta_i(u) = v$ for a path starting at q_i with input label u and output label v.

Example

$\Sigma = \{\alpha\}$
$A = \{a, b\}$



$\delta_0(\alpha^3) = aba$
$\delta_1(\alpha^3) = bab$

Doubling transducer

A *doubling transducer* is a transducer with set of states $\{q_0, q_1\}$ such that:

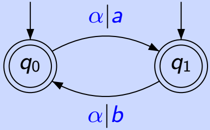
1. the input automata is a group automaton,
2. the output labels of the edges are all distinct.

$\delta_0, \delta_1 : \Sigma^* \to A^*$ are defined by $\delta_i(u) = v$ for a path starting at q_i with input label u and output label v.

The *image* of a set T is $\delta_0(T) \cup \delta_1(T)$.

Example

$\Sigma = \{\alpha\}$
$A = \{a, b\}$



$\delta_0(\alpha^3) = aba$
$\delta_1(\alpha^3) = bab$

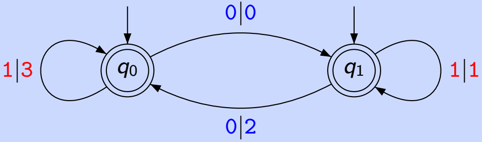
$\delta(\alpha^*) = (\varepsilon + a)(ba)^*(\varepsilon + b)$

G-palindromes

Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2016)]

Let S be a recurrent tree set **closed under reversal**.
The image of S by a doubling transducer is G-full, with $G \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

Example (doubling of Fibonacci)



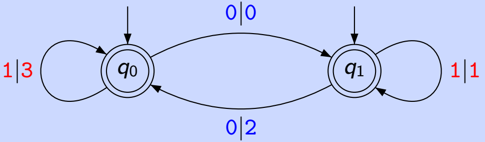
$$\{010\} \longrightarrow \{012,\ 230\}$$

G-palindromes

Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2016)]

Let S be a recurrent tree set **closed under reversal**.
The image of S by a doubling transducer is G-full, with $G \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

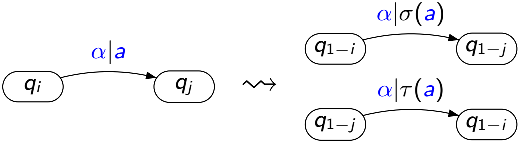
Example (doubling of Fibonacci)



σ : $0 \leftrightarrow 2$, $1 \leftrightarrow 3$

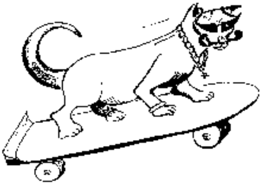
τ : $0, 2$ ⟲, $1 \leftrightarrow 3$

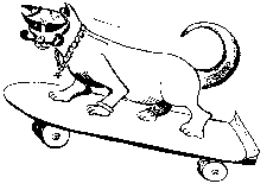
$\{010\} \longrightarrow \{012,\ 230\}$



$G = \{\mathrm{id}, \sigma, \tau, \sigma\tau\}$

MERCICREM





THANK YOUOY KNAHT