String attractors for factors of the Thue-Morse word

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String attractors

Definition [Kempa, Prezza (2018)]

A *string attractor* Γ of a word $u \in \mathscr{A}^+$ is a set of positions such that for every $w \in \mathscr{L}(u)$ there exists $j \in \Gamma$ s.t. $w = u_{j-\ell} \cdots u_j \cdots u_{j+r}$.

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Example

- $\underline{\mathtt{a}}\mathtt{na}\underline{\mathtt{n}}\mathtt{a}\underline{\mathtt{s}}$ $\quad \Gamma = \{1, 4, 6\}$

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- `ananas` $\Gamma = \{1,4,6\}$
- `pippi` $\Gamma = \{2,3\}$
- `lillagubben` $\Gamma = \{1,2, \ ,11\}$

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- banan $\Gamma_1 = \{1,3,4,5\}$
- banan $\Gamma_2 = \{1,2,3\}$

String attractors

A question of size

Proposition

A word u can have several string attractors. Moreover, for each such string attractor Γ

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Problem: Find a smallest string attractor for a given word. **NP-complete!**

Let $\gamma(u)$ denote the size of a smallest string attractor of $u \in \mathscr{A}^+$.

String attractors

Concatenation of words

Proposition [Mantaci, Restivo, Romana, Rosone (2021)]

Let $u, v \in \mathscr{A}^+$ with string attractors Γ_u and Γ_v. Then

$$\Gamma_{uv} = \Gamma_u \cup \{|u|\} \cup (\Gamma_v + |u|)$$

is a string attractor of uv.

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Example

$u = \underline{\mathtt{h}}\underline{\mathtt{e}}\underline{\mathtt{r}}\mathtt{r}, \quad v = \mathtt{n}\underline{\mathtt{i}}\underline{\mathtt{l}}\underline{\mathtt{s}}\mathtt{s}\underline{\mathtt{o}}\underline{\mathtt{n}} \qquad uv = \underline{\mathtt{h}}\underline{\mathtt{e}}\underline{\mathtt{r}}\underline{\mathtt{r}}\mathtt{n}\underline{\mathtt{i}}\underline{\mathtt{l}}\underline{\mathtt{s}}\mathtt{s}\underline{\mathtt{o}}\underline{\mathtt{n}}$

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Corollary

$$\gamma(uv) \leq \gamma(u) + \gamma(v) + 1$$

String attractors

Factors

Proposition [Mantaci, Restivo, Romana, Rosone (2021)]

The size of a smallest string attractor for a word is not a monotone measure, i.e., there exist $p, v, s \in \mathscr{A}^*$ s.t.

$$\gamma(pvs) < \gamma(v)$$

Example

$v = \mathtt{a}\underline{\mathtt{b}}\mathtt{b}\underline{\mathtt{b}}\mathtt{a}\underline{\mathtt{a}}\mathtt{b}, \quad \gamma(v) = 3$ but $v\mathtt{b} = \mathtt{abb}\underline{\mathtt{b}}\mathtt{a}\underline{\mathtt{a}}\mathtt{bb}, \quad \gamma(v\mathtt{b}) = 2$

String attractors and infinite words

Prefixes

An infinite word in general does not have a finite string attractor.

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$\mathbf{u}$

But we can consider $\mathrm{Pref}(\mathbf{u})$ (or at least a subset).

Theorem [Mantaci, Restivo, Romana, Rosone (2021)]

Every standard Sturmian word[⋆] has a minimal attractor of size 2.

$^\star s_0 = \mathrm{b}, \quad s_1 = \mathrm{a}, \quad s_{n+2} = s_{n+1}^{a_n} s_n$ where $(a_i)_{i\in\mathbb{N}}$ is the directive sequence of $\mathbf{s}$

String attractors and infinite words

String attractor profile

Definition [Schaeffer, Shallit (2021)]

The *string attractor profile* $s_{\mathbf{u}}$ of an infinite word $\mathbf{u}$ is defined as

$$s_{\mathbf{u}}(n) = \gamma(u_n)$$

where u_n is the prefix of length n of $\mathbf{u}$.

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Theorem [Schaeffer, Shallit (2021),]

- If $\mathbf{u}$ is linearly recurrent, then $s_{\mathbf{u}} = \Theta(1)$.
- If $\mathbf{u}$ is k-automatic, then $s_{\mathbf{u}} = \Theta(1)$ or $\Theta(\log n)$.

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Theorem [Schaeffer, Shallit (2021), Restivo, Romana, Sciortino (2022)]

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- If $\mathbf{u}$ is k-automatic, then $s_{\mathbf{u}} = \Theta(1)$ or $\Theta(\log n)$.
- If $\mathbf{u}$ is ultimately periodic, then $s_{\mathbf{u}} = \Theta(1)$.
- If $\mathbf{u}$ is a characteristic Sturmian word, then $s_{\mathbf{u}}(n) \leq 2$ for every n.

String attractor and infinite words

Factors

And if $u \in \mathscr{L}(\mathbf{u}) \setminus \mathrm{Pref}(\mathbf{u})$?

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Theorem [Dvořáková (2022)]

Let $\mathbf{u}$ be an episturmian sequence and $u \in \mathscr{L}(\mathbf{u})$. Then $\gamma(u) = |\mathrm{Alph}(u)|$.

Example

Let $\mathbf{w} = \mathtt{abacabaabacababacabaabacabacabaabacabab}\cdots$ be the Tribonacci word.

$$\underline{\mathtt{c}},\quad \mathtt{ab\underline{a}a\underline{b}a},\quad \mathtt{a\underline{c}aba\underline{a}bacaba\underline{b}acabaabacaba}$$

Thue-Morse

and some of its prefixes

The *Thue-Morse word* is the infinite word

$$\mathbf{t} = \texttt{abbabaabbaababbabaababbaabbabaab}\cdots$$

defined as $\mathbf{t} = \lim_{n\to\infty} t_n$ where

$$t_0 = \texttt{a} \quad \text{and} \quad t_{n+1} = t_n\overline{t_n}$$

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The *Prouhet-Thue-Morse word* is the infinite word

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$t_4 = \mathtt{abbabaabbaababba}$
...

Note that $|t_n| = |\overline{t_n}| = 2^n$.

String attractor of prefixes of Thue-Morse

Theorem [Mantaci, Restivo, Romana, Rosone (2019)]

For any $n \geq 3$ we have $\gamma(t_n) \leq n$. In particular, a string attractor for t_n is given by

$$\{2^{n-1}+1\} \bigcup_{i=2}^{n} \{3 \cdot 2^{i-2}\}$$

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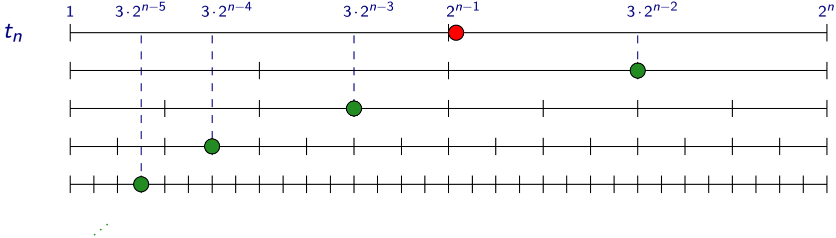

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Example

$$t_5 = \text{ab}\underline{\text{b}}\text{ab}\underline{\text{a}}\text{abbaa}\underline{\text{b}}\text{abba}\underline{\text{b}}\text{aababb}\underline{\text{a}}\text{abbabaab}$$

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Conjecture [Mantaci, Restivo, Romana, Rosone (2019)]

For every large enough n we have $\gamma(t_n) = n$.

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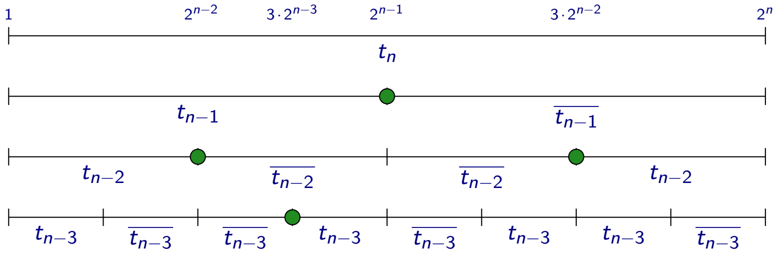
Theorem [Kutsukake, Matsumoto, Nakashina, Bannai, Takeda (2020)]

For any $n \geq 4$ we have $\gamma(t_n) = 4$. Moreover, a smallest string attractor for t_n is given by

$$\Gamma_n := \{2^{n-2},\quad 3 \cdot 2^{n-3},\quad 2^{n-1},\quad 3 \cdot 2^{n-2}\}$$



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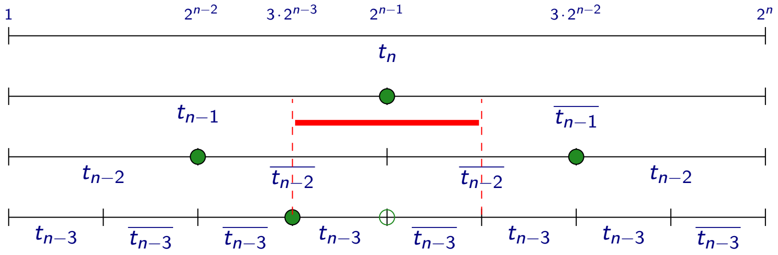

Idea: $t_k \cdot \overline{t_k} = t_{k-1}\overline{t_{k-2}}t_{k-2} \cdot \overline{t_{k-2}}t_{k-2}t_{k-1}$

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Theorem [D.]

For any $u \in \mathscr{L}(\mathbf{t})$ we have $\gamma(u) \leq 5$.

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[*A vague and confused idea of the*] **Proof:**

- For $u \in \mathscr{L}(t_n)$ with $n \leq 5$, just check by brute force.

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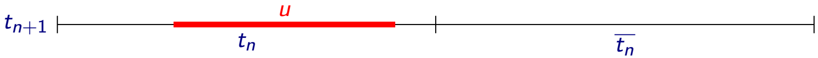
- For $u \in \mathscr{L}(t_{n+1}) = \mathscr{L}(t_n \overline{t_n})$, with $n > 5$.

  $\rightarrow$ If $u \in \mathscr{L}(t_n) \cup \mathscr{L}(\overline{t_n})$, by induction.



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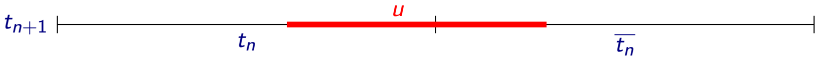
- For $u \in \mathscr{L}(t_{n+1}) = \mathscr{L}(t_n \overline{t_n})$, with $n > 5$.

  $\rightarrow$ If u contains the center of $t_n \overline{t_n}$, then let us consider *a few* $(72+)$ cases



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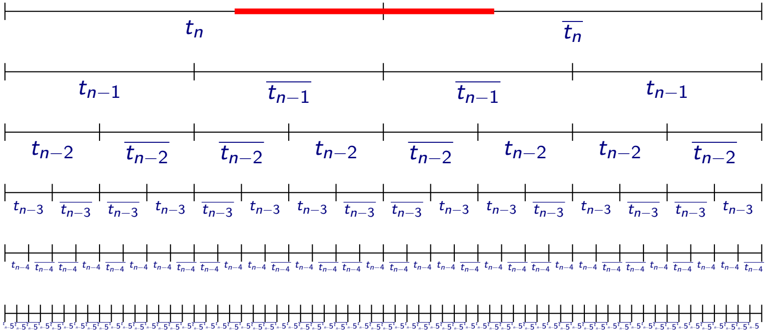

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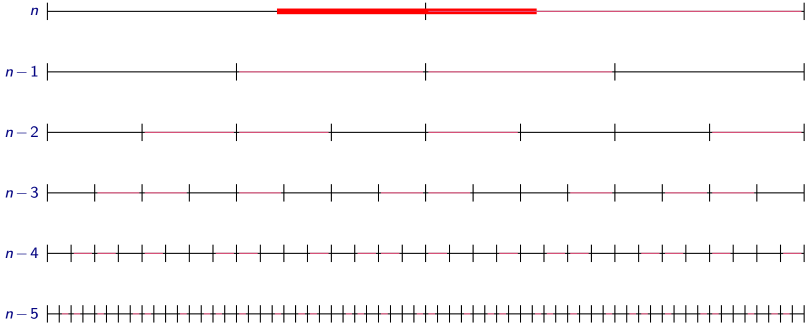

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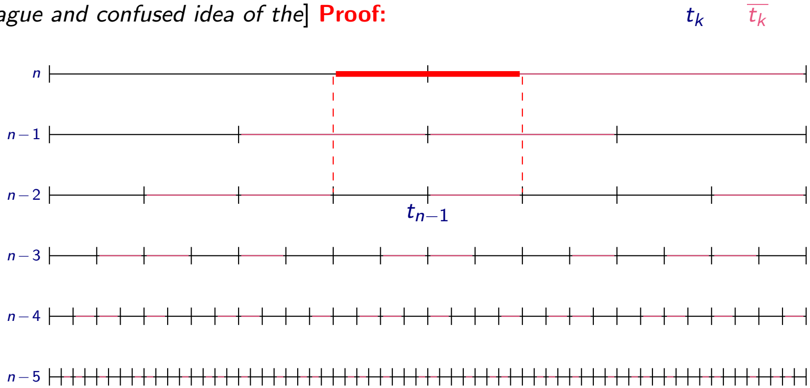

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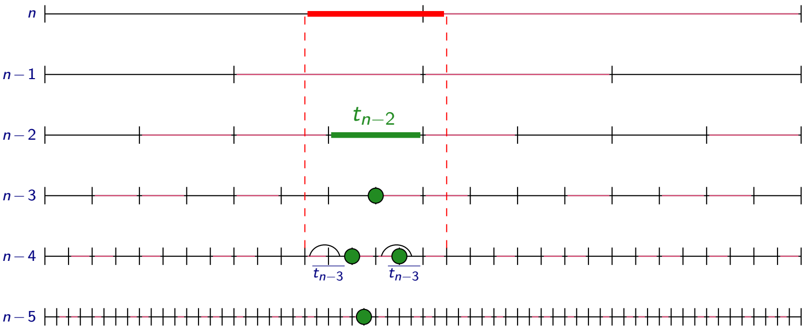

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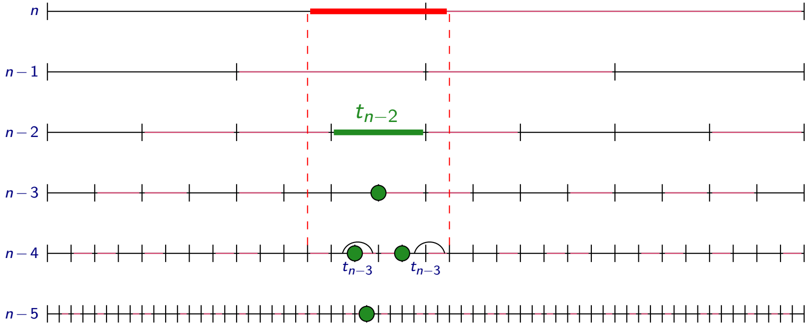

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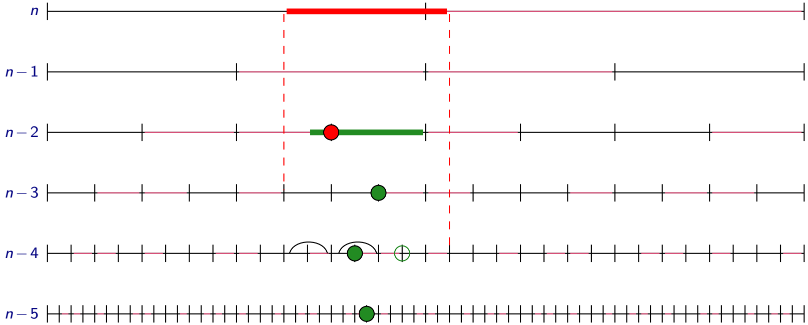

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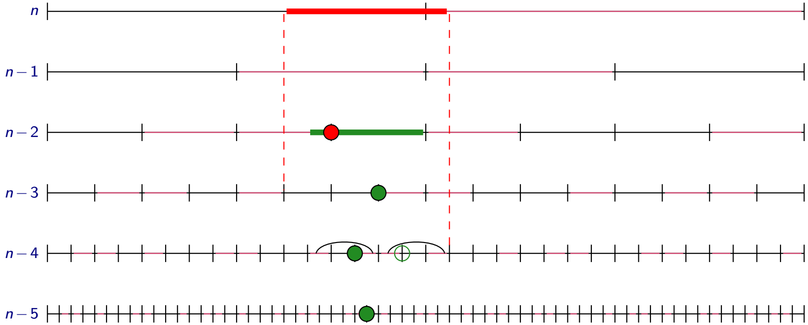

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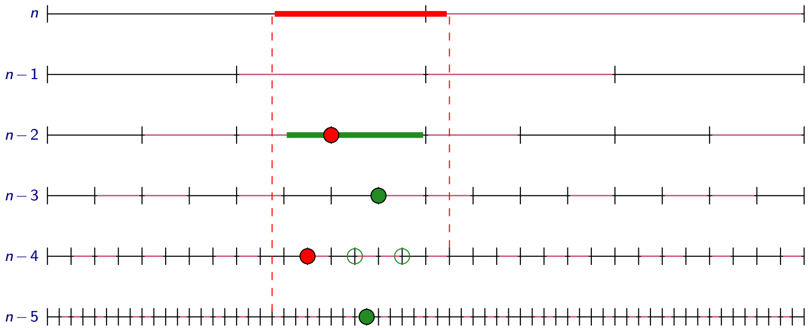

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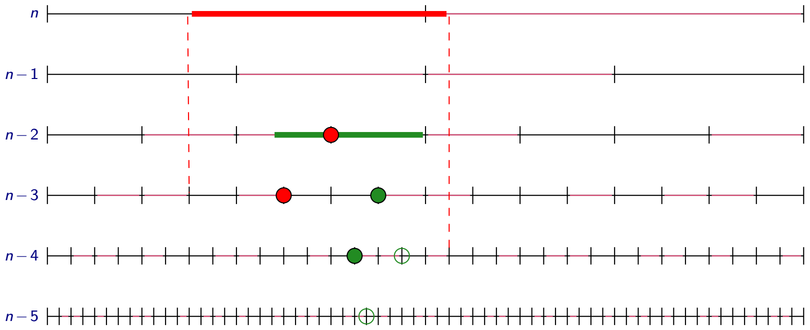

String attractor for factors of Thue-Morse

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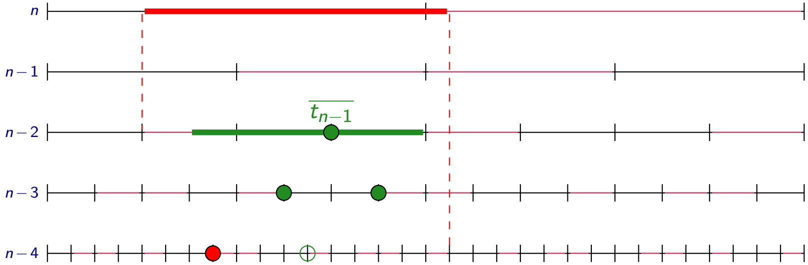

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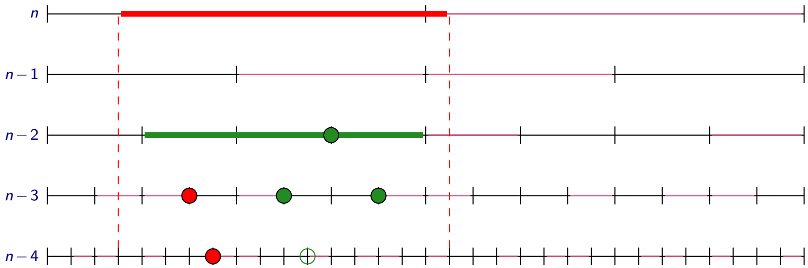

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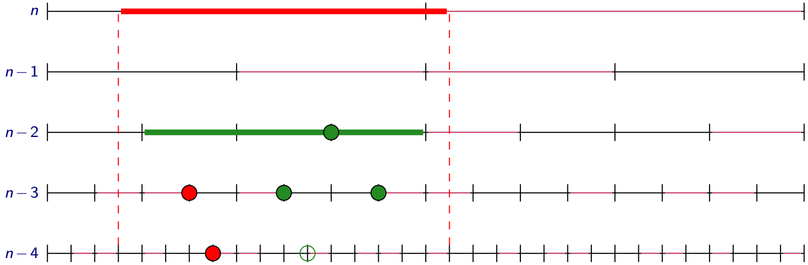

$w = \mathrm{b}\,\overline{t_3}\,\overline{t_4}\,\mathrm{b} = \mathrm{bbaa\underline{b}a\underline{b}babaa\underline{b}abba\underline{a}bba\underline{b}aabb} \in \mathscr{L}(t_6)$

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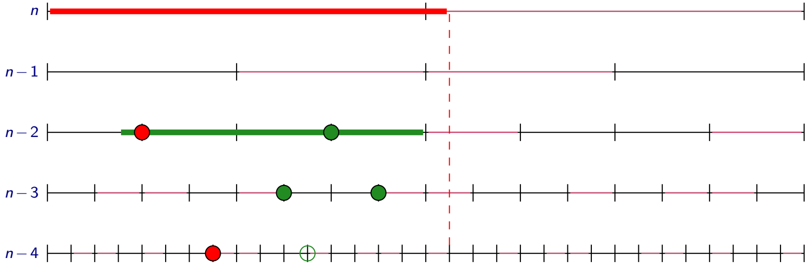

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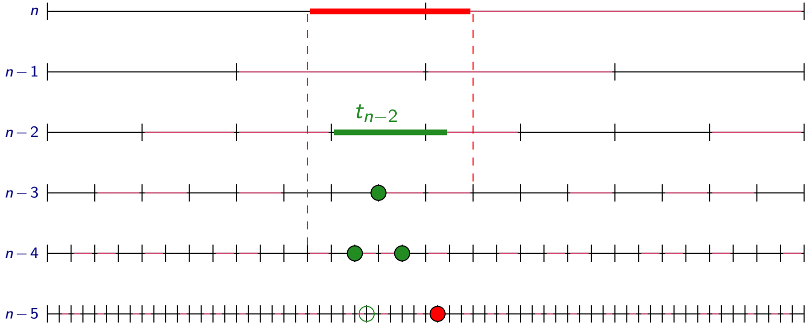

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And 28 other similar cases + all the symmetrical ones...

Future works

- Generalized Thue-Morse word over $\{\mathtt{a}_1, \mathtt{a}_2, \dots, \mathtt{a}_\mathtt{m}\}$

$$\mathbf{t}_m = \lim_{n\to\infty} \varphi_m^n(\mathtt{a}_1)) \qquad \text{where} \quad \varphi_m(\mathtt{a}_\mathtt{k}) = \mathtt{a}_\mathtt{k} \cdots \mathtt{a}_\mathtt{m}\, \mathtt{a}_1 \cdots \mathtt{a}_{\mathtt{k}-1}$$

Conjecture

There exists an integer K_m such that $\gamma(u) \leq K_m$ for every $u \in \mathscr{L}(\mathbf{t}_m)$.

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- Use different techniques (e.g., pseudo-palindromic closure)

Theorem [Dvořáková (2022)]

A minimal string attractor for a non-empty palindromic prefix u of a standard episturmian word is given by $\left\{ \max\{|p| \,:\, p = \tilde{p} \quad \text{and} \quad pa \in \mathrm{Pref}(u)\} \right\}_{a \in \mathrm{Alph}(u)}$

$$\mathbf{t} = \psi_{(ER)^\omega}(\mathtt{ab}^\omega) \qquad (\ \mathtt{ab}, \mathtt{abba}, \mathtt{abbabaab}, \mathtt{abbabaabbaababba}, \ldots\ )$$

Tack för er uppmärksamheten