Maximal bifix decoding

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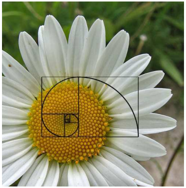
Fibonacci

$$x = abaababaabaababa \cdots$$

$$x = \lim_{n\to\infty} \varphi^n(a) \qquad \text{where} \qquad \varphi : \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$



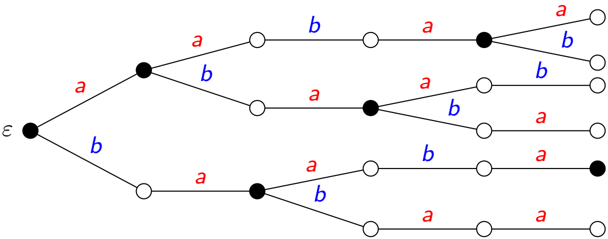
Fibonacci

$$x = abaababaabaababa\cdots$$

The *Fibonacci set* (set of factors of x) is a Sturmian set.

Definition

A *Sturmian* set S is a factorial set such that $p_n = \text{Card}\,(S \cap A^n) = n + 1$.



n	0 ,	1 ,	2 ,	3 ,	4 ,	5 ,	$\cdots$
p_n :	1 ,	2 ,	3 ,	4 ,	5 ,	6 ,	$\cdots$

2-coded Fibonacci

$$x = ab\ aa\ ba\ ba\ ab\ aa\ ba\ ba\ \cdots$$

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Arnoux-Rauzy sets

Definition

An *Arnoux-Rauzy* set is a factorial set closed by reversal with $p_n = (\mathrm{Card}\,(A) - 1)n + 1$ having a unique right special factor for each length.

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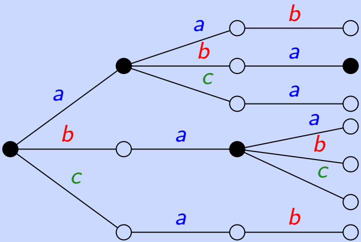
Example (Tribonacci)

Factors of the fixed point $\psi^\omega(a)$ of the morphism $\quad \psi : a \mapsto ab, \quad b \mapsto ac, \quad c \mapsto a.$



2-coded Fibonacci

$$f^{-1}(x) = v\ u\ w\ w\ v\ u\ w\ w\ \cdots$$

Is the set of factors of $f^{-1}(S)$ an Arnoux-Rauzy set ?

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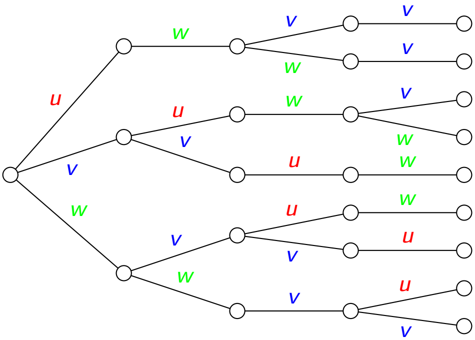

n	0 ,	1 ,	2 ,	3 ,	4 ,	$\cdots$
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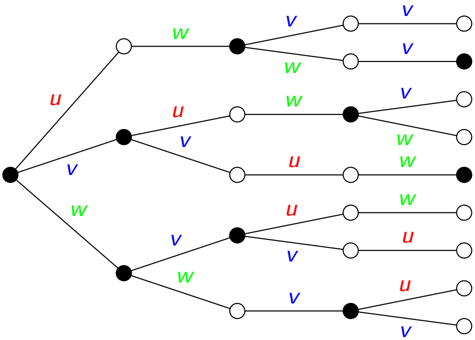
2-coded Fibonacci

$$f^{-1}(x) = \; v\; u\; w\; w\; v\; u\; w\; w\; \cdots$$

Is the set of factors of $f^{-1}(S)$ an Arnoux-Rauzy set ? No !



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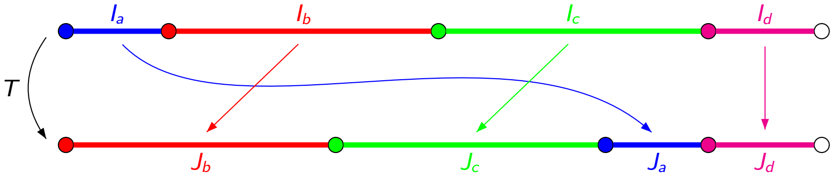
Interval exchanges

Let $(I_\alpha)_{\alpha \in A}$ and $(J_\alpha)_{\alpha \in A}$ be two partitions of $[0,1[$.
An *interval exchange transformation* (IET) is a map $T : [0,1[\to [0,1[$ defined by

$$T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha.$$



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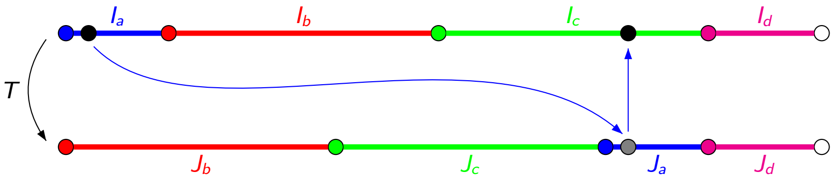

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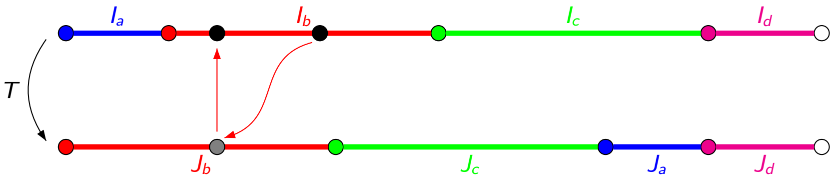

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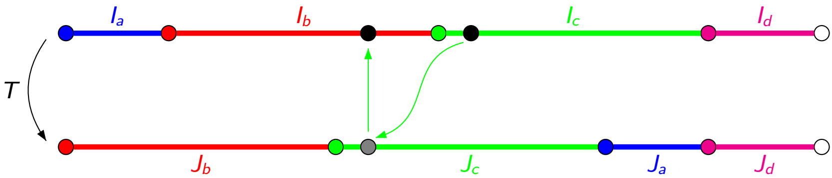

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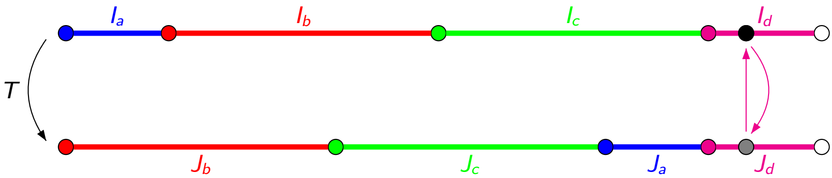

Interval exchanges

T is said to be *minimal* if for any point $z \in [0, 1[$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[0, 1[$.

T is said *regular* if the orbits of the separation points $\neq 0$ are infinite and disjoint.

Theorem [M. Keane (1975)]

A regular interval exchange transformation is minimal.

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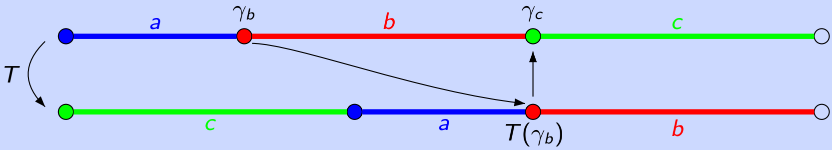
Example (the converse is not true)

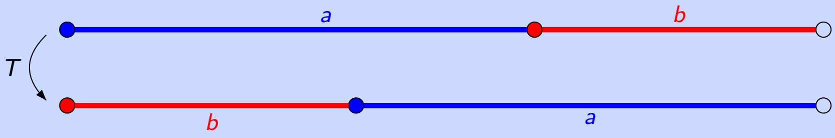


Interval exchanges

The *natural coding* of T relative to $z \in [0,1[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in A^\omega$ defined by

$$a_n = \alpha \quad \text{if } T^n(z) \in I_\alpha .$$

Example (Fibonacci, $z = (3-\sqrt{5})/2$)



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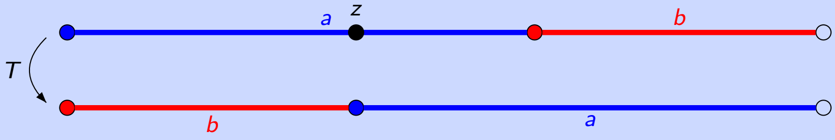

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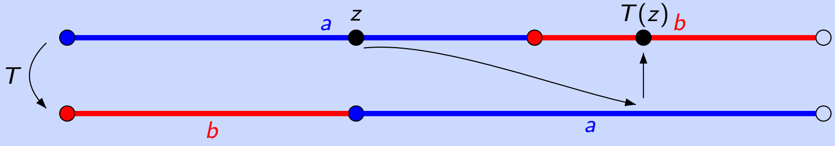

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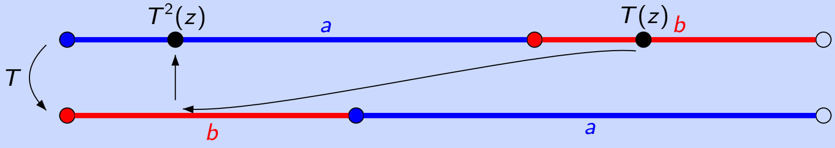

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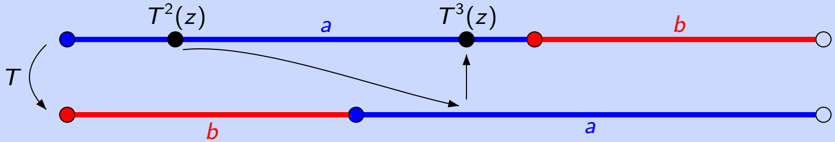

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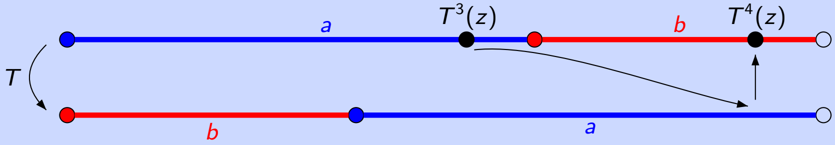

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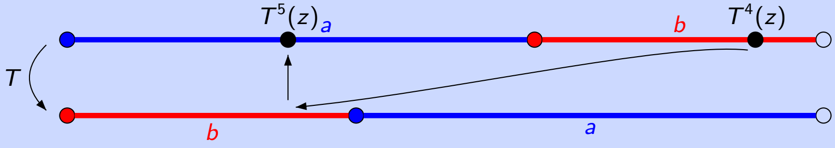

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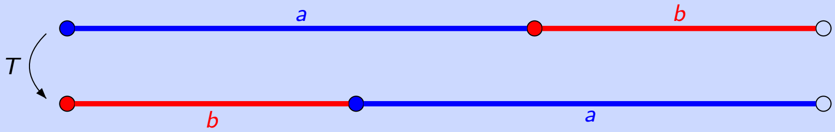
Interval exchanges

The set $\mathcal{L}(T) = \bigcup_{z\in[0,1[} \mathrm{Fac}(\Sigma_T(z))$ is said a (*minimal*, *regular*) *interval exchange set*.

Remark. If T is minimal, $\mathrm{Fac}(\Sigma_T(z))$ does not depend on the point z.

Example (Fibonacci)



$$\mathcal{L}(T) = \left\{\varepsilon, a, b, aa, ab, ba, aab, aba, baa, \dots\right\}$$

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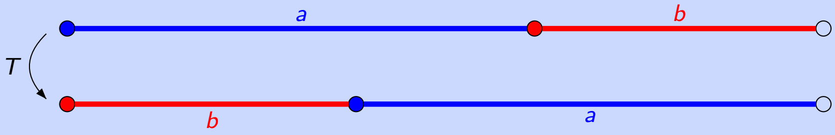

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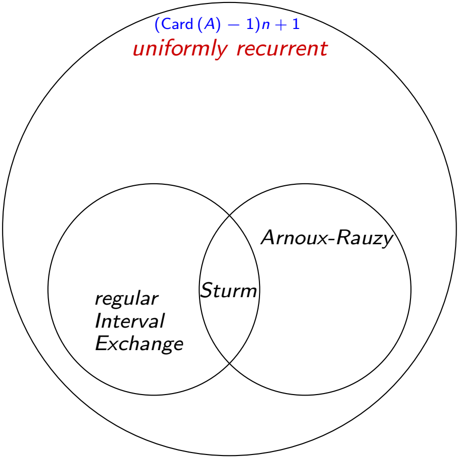
Proposition

Regular interval exchange sets have factor complexity $p_n = (\mathrm{Card}(A) - 1)n + 1$.

Arnoux-Rauzy and Interval exchanges



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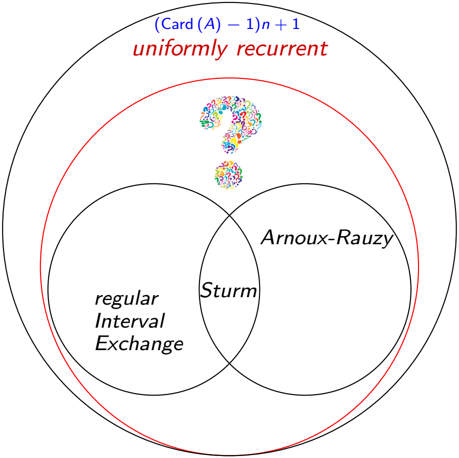

Extension graphs

The *extension graph* of a word $w \in S$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{a \in A \,|\, aw \in S\}, \\ R(w) &= \{a \in A \,|\, wa \in S\}, \\ B(w) &= \{(a,b) \in A \,|\, awb \in S.\} \end{aligned}$$

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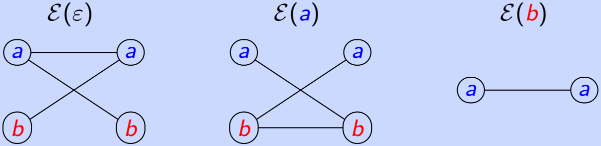

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The *multiplicity* of a word w is the quantity

$$m(w) = \text{Card}\,(B(w)) - \text{Card}\,(L(w)) - \text{Card}\,(R(w)) + 1.$$

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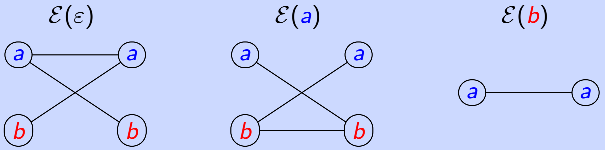
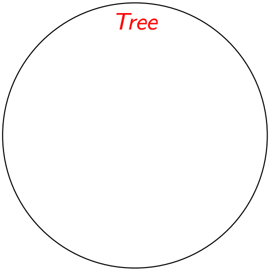


Tree and neutral sets

Definition

A factorial set S is called a *tree set* if the graph $\mathcal{E}(w)$ is a tree for any nonempty $w \in S$.



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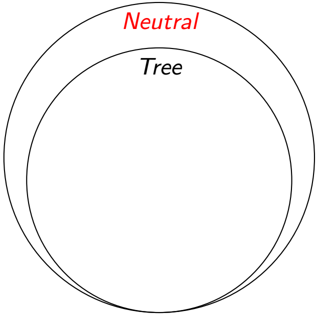

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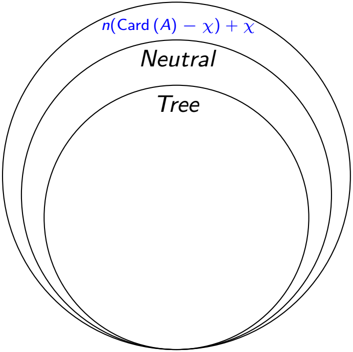
The *characteristic* of a neutral/tree set S is the quantity $\chi(S) = 1 - m(\varepsilon)$.



[*using* J. Cassaigne : "**Complexité et facteurs spéciaux**" (1997).]

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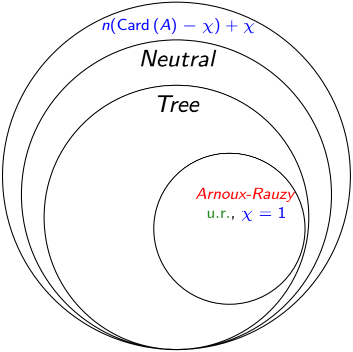

[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone : "**Acyclic, connected and tree sets**" (2014).]

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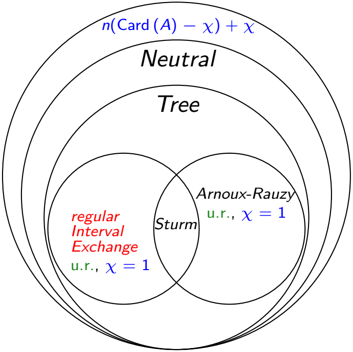

[Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone : "**Bifix codes and interval exchanges**" (2015).]

Recurrence and uniformly recurrence

Definition

A factorial set S is *recurrent* if for every $u, v \in S$ there is a $w \in S$ such that uvw is in S.

It is *uniformly recurrent* (or *minimal*) if for every $u \in S$ there exists an $n \in \mathbb{N}$ such that u is a factor of every word of length n in S.

Proposition

Uniform recurrence $\Longrightarrow$ recurrence.

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Theorem [D., Perrin (2016)]

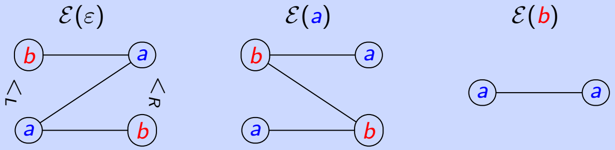
A recurrent neutral set is uniformly recurrent.

Planar tree sets

Let $<_L$ and $<_R$ be two orders on A.
For a set S and a word $w \in S$, the graph $\mathcal{E}(w)$ is *compatible* with $<_L$ and $<_R$ if for any $(a, b), (c, d) \in B(w)$, one has

$$a <_L c \quad \Longrightarrow \quad b \leq_R d.$$

Example (Fibonacci, $a <_L b$ and $b <_R a$)

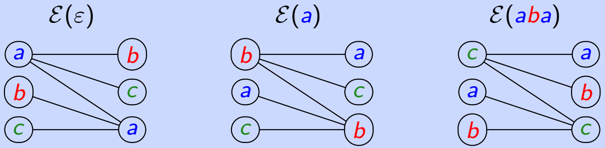


A biextendable set S is a *planar tree set* w.r.t. $<_L$ and $<_R$ on A if for any nonempty $w \in S$ (resp. ε) the graph $\mathcal{E}(w)$ is a tree (resp. forest) compatible with $<_L$ and $<_R$.

Planar tree sets

Example

The *Tribonacci set* is **not** a planar tree set.
Indeed, let us consider the extension graphs of the bispecial words ε, a and aba.



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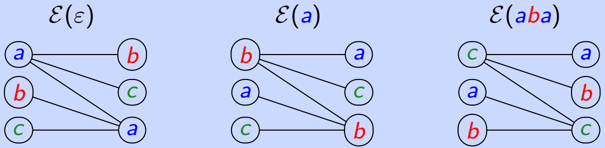
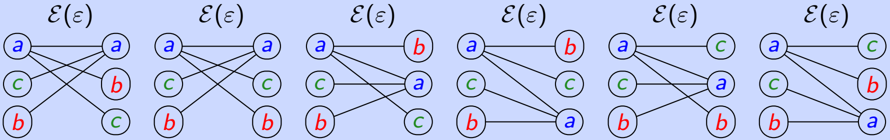


- $a <_L c <_L b$



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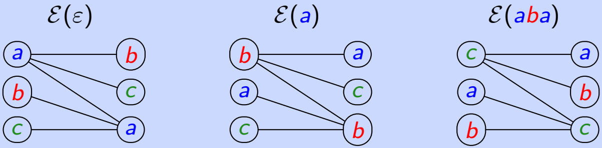
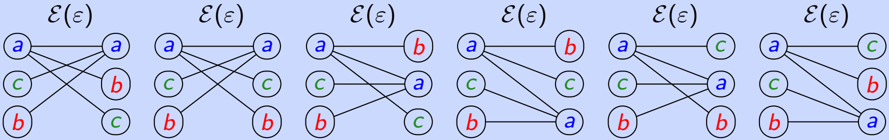


- $\underline{a <_L c <_L b}$ $\Longrightarrow$ $b <_R c <_R a$ or $c <_R b <_R a$



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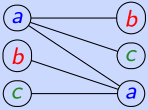
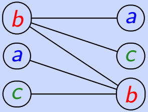
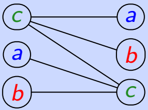
$\mathcal{E}(\varepsilon)$



$\mathcal{E}(a)$

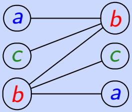


$\mathcal{E}(aba)$



- $\underline{a <_L c <_L b}$ $\quad\Longrightarrow\quad$ $b <_R c <_R a$ or $c <_R b <_R a$

$\mathcal{E}(a)$



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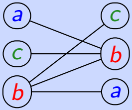
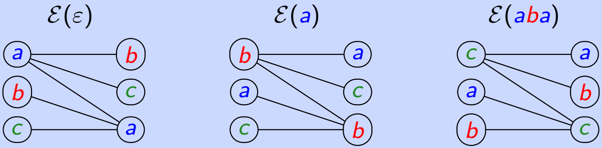


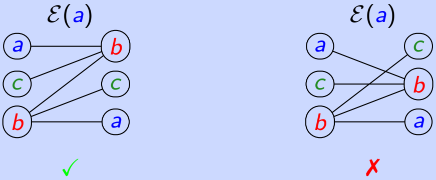
Planar tree sets

Example

The *Tribonacci set* is **<u>not</u>** a planar tree set.
Indeed, let us consider the extension graphs of the bispecial words ε, a and aba.



- <u>$a <_L c <_L b$</u> $\Longrightarrow$ $b <_R c <_R a$



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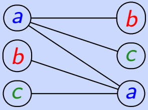

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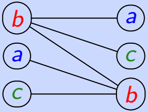

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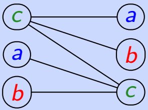
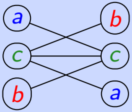


- $a <_L c <_L b \quad \Longrightarrow \quad b <_R c <_R a$

$\mathcal{E}(aba)$



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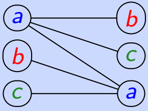
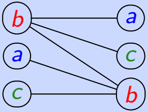


$\mathcal{E}(a)$



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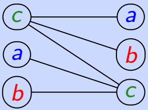

- $\underline{a <_L c <_L b}$ $\Longrightarrow$ ↯

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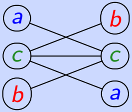
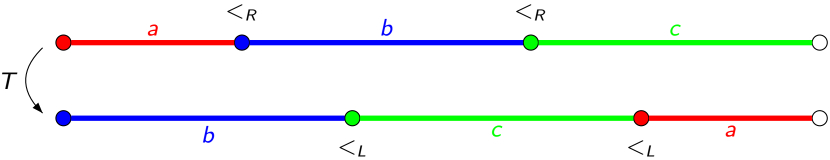


✗

Planar tree sets

Theorem [S. Ferenczi, L. Zamboni (2008)]

A set S is a regular interval exchange set on A if and only if it is a recurrent planar tree set of characteristic 1.



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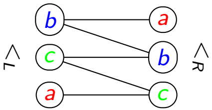
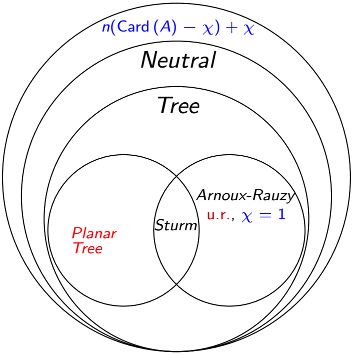


Tree and neutral sets



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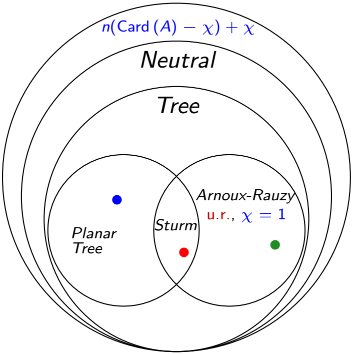
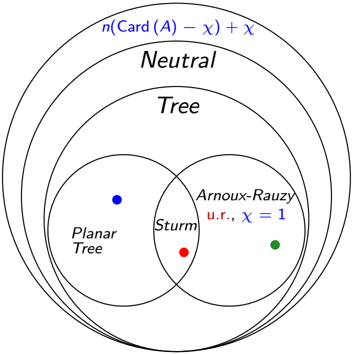


- Fibonacci
- Tribonacci
- regular IET $(\mathrm{Card}\,(A) \geq 3)$

Tree and neutral sets



- ● Fibonacci
- ? 2-coded Fibonacci
- ● Tribonacci
- ? 2-coded Tribonacci
- ● regular IET (Card $(A) \geq 3$)
- ? 2-coded regular IET

Bifix codes

Definition

A *bifix code* is a set $X \subset A^+$ of nonempty words that does not contain any proper prefix or suffix of its elements.

Example

- ✓ $\{aa, ab, ba\}$
- ✓ $\{aa, ab, bba, bbb\}$
- ✓ $\{ac, bcc, bcbca\}$
- ✗ { fire, water, Waterloo }
- ✗ { false, Montreal, real }
- ✗ { onto, toro, Toronto }

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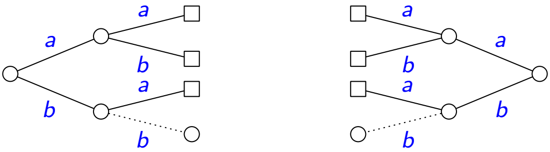
A *bifix code* is a set $X \subset A^+$ of nonempty words that does not contain any proper prefix or suffix of its elements.

A bifix code $X \subset S$ is S-*maximal* if it is not properly contained in a bifix code $Y \subset S$.

Example (Fibonacci)

The set $X = \{aa, ab, ba\}$ is an S-maximal bifix code.
It is not an A^*-maximal bifix code, since $X \subset X \cup \{bb\}$.



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A *coding morphism* for a bifix code $X \subset A^+$ is a morphism $f : B^* \to A^*$ which maps bijectively B onto X.

Example

The map $f : \{u, v, w\}^* \to \{a, b\}^*$ is a coding morphism for $X = \{aa, ab, ba\}$.

$$f : \begin{cases} u \mapsto aa \\ v \mapsto ab \\ w \mapsto ba \end{cases}$$

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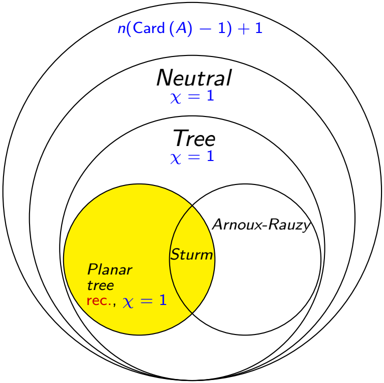
When S is factorial and X is an S-maximal bifix code, the set $f^{-1}(S)$ is called a *maximal bifix decoding* of S.

Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014)]

The family of recurrent planar tree sets of characteristic 1 (i.e. regular interval exchange sets) is closed under maximal bifix decoding.



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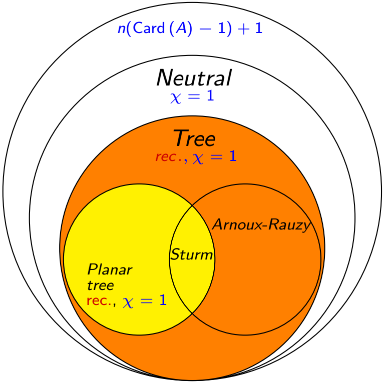
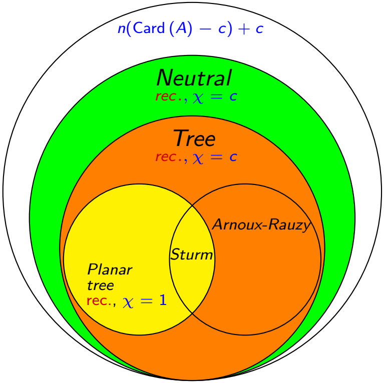


Maximal bifix decoding

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014, 2015) ; D., Perrin (2016)]

The family of recurrent neutral sets (resp. tree sets) of characteristic c is closed under maximal bifix decoding.



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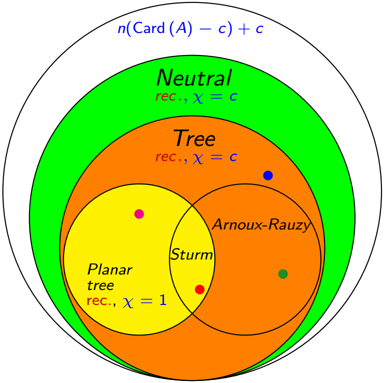


- Fibonacci
- 2-coded Fibonacci
- Tribonacci
- 2-coded Tribonacci

Parse and degree

Definition

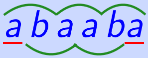
A *parse* of a word w with respect to a bifix code X is a triple (q, x, p) such that :

- $w = qxp$,
- q has no suffix in X,
- $x \in X^*$ and
- p has no prefix in X.

Example

Let $X = \{aa, ab, ba\}$ and $w = abaaba$. The two possible parses of w are :

- $(\varepsilon, ab\,aa\,ba, \varepsilon)$,
- $(a, ba\,ab, a)$.



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The S-*degree* of X is the maximal number of parses with respect to X of a word of S.

Example (Fibonacci)

- The set $X = \{aa, ab, ba\}$ has S-degree 2.
- The set $X = S \cap A^n$ has S-degree n.

Cardinality of bifix codes

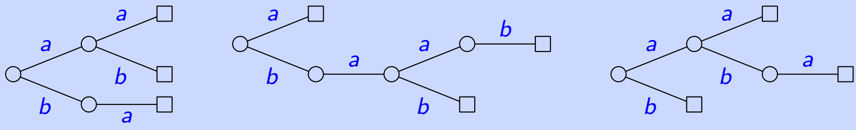
Theorem [D., Perrin (2016)]

Let S be a recurrent neutral set of characteristic c.
For any finite S-maximal bifix code X of S-degree n, one has

$$\mathrm{Card}\,(X) = n\big(\mathrm{Card}\,(A) - c\big) + c.$$

Example (Fibonacci)

The three possible S-maximal bifix codes of S-degree 2 are :



Each of them has cardinality $3 = 2(2-1)+1$.

Cardinality of bifix codes

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Let S be a recurrent neutral set of characteristic c.
For any finite S-maximal bifix code X of S-degree n, one has

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Theorem [D., Perrin (2016)]

Let S be a uniformly recurrent set.
If every finite S-maximal bifix code of S-degree n has $n\big(\mathrm{Card}\,(A) - c\big) + c$ elements, then S is neutral of characteristic c.

Finite index basis property

Example (Fibonacci)

The S-maximal bifix code $X = \{aa, ab, ba\}$ of S-degree 2 is a basis of $\langle A^2 \rangle$. Indeed

$$bb = ba\,(aa)^{-1}\,ab$$

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The S-maximal bifix code $X = \{aa, ab, ba\}$ of S-degree 2 is a basis of $\langle A^2 \rangle$. Indeed

$$bb = ba\,(aa)^{-1}\,ab$$

Also $S \cap A^3 = \{aab, aba, baa, bab\}$ is a basis of $\langle A^3 \rangle$:

$$\begin{array}{rcl} aaa & = & aab(bab)^{-1}baa \\ abb & = & aba(baa)^{-1}bab \\ bba & = & bab(aab)^{-1}aba \\ bbb & = & bba(aba)^{-1}aab \end{array}$$

Finite index basis property

Definition

A set $S \subset A^+$ satisfies the *finite index basis property* if for any finite bifix code $X \subset S$:

X is an S-maximal bifix code of S-degree d if and only if it is a basis of a subgroup of index d of the free group on A.

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Theorem [Berstel, De Felice, Perrin, Reutenauer, Rindone (2012)]

An Arnoux-Rauzy set satisfies the finite index basis property.

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Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014)]

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Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

A (uniformly) recurrent tree set of characteristic 1 satisfies the finite index basis property.

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Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

A uniformly recurrent set satisfying the finite index basis property is a tree sets of characteristic 1.

Further research directions

and some open problem

- Decidability of the tree condition

  [work in progress with Revekka Kyriakoglou and Julien Leroy]

- Tree sets and palindromes

  [S tree of $\chi = 1$ closed under reversal $\implies$ S full $\left(\implies \mathrm{Pal}(n) = \begin{cases} 1 & \text{odd} \\ |A| & \text{even} \end{cases} \right)$]

- Return words are a basis of the free group

  [S recurrent tree of $\chi = 1 \quad \implies \quad \mathcal{R}_S(w)$ basis of $\mathbb{F}_A$ for all $w \in S$]

- Rigidity of tree sets

  [stabilizer of a *tree word*.]

Thank you

Merci